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The Footloose Entrepreneur Model with Three Regions

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Master degree dissertation in Economics

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Biography

José Maria Lopes Gaspar was born in Lisbon on 28 September of 1988.

He lived in his hometown until 2006, after which he moved to Oporto to get his Bachelor (BSc) degree in Economics at the School of Economics and Management (FEP) of the University of Porto. He graduated in 2010 with a final average of fourteen (14) points out of twenty (20).

During this time, he was part of the University of Porto's team of the student organization *Sharing Knowledge*, where he participated in the development of projects and presentations in the field of macroeconomics.

Immediately after the BSc, he joined the Master of Science (MSc) in Economics at that same institution. Upon completion of the curricular part of the MSc, his average was of eighteen (18) points out of twenty (20). Just prior to starting to work on his dissertation, he tutored students from the portuguese-speaking african countries (PALOP) in support classes of Macroeconomics I at FEP.

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Dedico esta dissertação ao meu pai, Alfredo.

Abstract

We study an analytically solvable version of Krugman's Core-Periphery model extended to three regions. This is the 3-region Footloose Entrepreneur model based on the 2-region version by Forslid and Ottaviano (2003). Solvability is achieved by employment of the skilled inter-regionally mobile work-force in the fixed costs and the unskilled immobile work-force in the variable costs of the manufacturing firms. This enables us to obtain real wages as explicit functions of the spatial distributions of skilled workers in the three regions. The main aim of this work is to study long-run equilibria of the model in terms of skilled workers' migration, which is based on indirect utility differentials as given by differences in inter-regional real wage differences. For this purpose, we focus on the analysis of stability of three types of equilibria: concentration, total and partial dispersion. The first corresponds to full agglomeration of industry in one region, the second refers to an equalized spatial distribution of skilled workers across the three regions and the third pertains to a situation whereby skilled workers are equally dispersed in two of the three regions. We show analytical and numerical evidence in that the latter configuration is always unstable. We also compare our results concerning concentration and total dispersion with those of the 2-region model. Finally, we discuss the existence and robustness of bifurcations in our 3-region model.

Keywords: new economic geography, core-periphery, footloose entrepreneur

JEL Classification Numbers: R10, R12, R23

Resumo

Estudamos uma versão analiticamente resolúvel do modelo “core-periphery” de Krugman estendido para três regiões. Trata-se do modelo “Footloose Entrepreneur” com três regiões baseado na versão com duas regiões por Forslid e Ottaviano (2003). A solvabilidade do modelo é alcançada através da incorporação da força de trabalho qualificada e inter-regionalmente móvel nos custos fixos e da força de trabalho imóvel e não qualificada nos custos variáveis das empresas industriais. Isto permite-nos obter os salários reais como funções explícitas da distribuição espacial dos trabalhadores qualificados nas três regiões. O objetivo principal deste trabalho é o de estudar os equilíbrios de longo prazo em termos da migração de trabalhadores qualificados, que é baseada em diferenciais de utilidade indireta dados pelas diferenças inter-regionais dos salários reais. Para este fim, concentramo-nos na análise da estabilidade de três tipos de equilíbrio: concentração, dispersão total e parcial. O primeiro corresponde à aglomeração total da indústria numa região, o segundo refere-se a uma distribuição espacial dos trabalhadores qualificados equalizada pelas três regiões, e o terceiro refere-se a uma situação em que os trabalhadores qualificados estão igualmente dispersos em duas das três regiões. Apresentamos evidência analítica e numérica no sentido de que a última configuração seja sempre instável. Comparamos também os nossos resultados, relativos à concentração e à dispersão total, com as do modelo para duas regiões. Finalmente, discutimos a existência e robustez de bifurcações no modelo com três regiões.

Palavras-chave: nova economia geográfica, centro-periferia, footloose entrepreneur

Código JEL: R10, R12, R23

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1 Introduction

The secular tendency for agglomeration of economic activity in specific industrial sectors in many countries and regions is well known and has been a matter of profound debate for a long time. However, up to recent years, the economic science had either neglected or simply failed to explain the spatial distribution of the factors of production and economic activity in general. If not due to a disregard for its evident importance and its consequences on the unequal income distributions, increasing clustering of economic activity and integration, then at least doubtlessly due to the difficulties it imposes on the analytical treatment of increasing returns, as is suggested by Fujita *et al.* (1999). The insistent clinging to constant returns as a simplifying assumption makes it unfeasible to explain the cumulative process of the self-perpetuating geographic concentration of economic activities. Howbeit, in the past few decades, there have been a lot of theoretical advances in this field of Economic Geography, partially thanks to new techniques developed in economic modeling, numerical methods and computation, specially in other fields such as industrial economics and economic growth. These allowed for a more rigorous treatment, based on microeconomic foundations, of the phenomena associated with the agglomeration of economic activities.

Economic Geography can be understood as the study of where the different economic activities take place and the intrinsic reasons that justify it. Transport costs, increasing returns and factor mobility are among such reasons. Of course, it would be unfair to claim that such efforts in trying to explain industrial location have not been made in the somewhat distant past, like the development of models that descend from the von Thünen theory (von Thünen, 1826) on agricultural land use.

Johann von Thünen was arguably a pioneer in addressing the causes that led to spatial location and how this would affect prices, before industrialization had occurred. In his famous model, different types of agriculture were placed in four concentric rings around a central city. Close proximity to the city meant low transport costs, so the farther the rings were from the city, the more agricultural products not susceptible to high transport costs were produced.[1] Inversely, land cost was highest in the ring surrounding the city, being a decreasing function of the distance to the city. The model, thus, established an inverse relation between rent and distance from the center of the city centre and, hence, spatial location.

Nevertheless, it was not until recent years that Economic Geography jumped into mainstream economics, and it has been gaining more and more relevance ever since. Under this premise, many New Economic Geography (NEG) models have been developed. Among these, that which may be considered the benchmark of NEG models is the Core-Periphery (CP) model with two regions by Krugman (1991b), a model based on Dixit-Stiglitz (1977) monopolistic competition. In particular, this model explains geographic concentration due to the increasing economic integration, discerning the different forces that lead to or drive away from agglomeration outcomes. The spatial distribution (and therefore concentration) of the economy depends on labour migration which in turn responds to real wage differences across different regions.[2]

Three main effects drive the mechanics of the CP model. First, there is a “market size effect”, whereby a new entrant firm in a market causes local expenditures to grow and generates a positive impact on demand per firm, thus implying a preference for bigger markets on behalf of the firms. The second effect is called “cost of living” effect and stems from a decrease in price indices induced by the increase of competition due to the entrance of a new firm which results in a positive impact on consumer surplus. Specifically, this price reduction reflects the narrower array of goods that consumers have to import from abroad and that are subject to transport costs. Finally, we have the “market crowding effect”, by which the aforementioned decrease in price indexes naturally has a negative impact on demand per firm. The first two encourage full agglomeration outcomes, while the third clearly discourages geographic concentration.

It is widely acknowledged that the original NEG model suffers from the fact that it

[1] We can think of livestock, since animals are able to transport themselves, thus implying lower transport costs. On the other hand, perishable goods that needed to be transported quickly would be produced closer to the city.

[2] The original CP model by Krugman (1991b) exhibits no explicit dynamics. These are shown in the monograph by Fujita *et al.* (1999).

has no analytic solution. Nonetheless, extensive literature has been able to derive interesting and relevant conclusions, whether analytical or resulting from numerical simulations. The synopsis in Fujita *et al.* (1999) is widely acknowledged as a bulwark in NEG, and NEG models have come to rely heavily on the modeling tricks and results summarized by it. More recent works have focused on further enhancing the possibility of deriving results from existing NEG models, like those from Robert-Nicoud (2005), Mossay (2006), and the possibility to relate them with other subjects, as in Fujita and Thisse (2003) where the authors combine Krugman's CP model with an endogenous growth model to study the effects of agglomeration on economic growth.

From the difficulty of dealing with Krugman's CP model, the need arises of modifications that make it easier to manipulate analytically, enabling us to extract more from the model than just a gallery of possible outcomes. For instance, in Ottaviano *et al.* (2002), the assumptions on trade costs and utility functions of Krugman's CP model are changed to account for the possibility of deriving analytical results. While this renders the solutions obtained by Krugman (1991b) analytically, some of the main features of the original CP model have been dropped off. These include changes on assumptions on preferences and transport costs.

That being said, there is still a need for modifications that do not change its fundamentals, while making it much more tractable. Following this line of thought, Forslid and Ottaviano (2003) have proposed an analytically solvable version of the CP model with two regions, with skill heterogeneity between agricultural and industrial workers and restricted inter-regional mobility to the more skilled workers. They employ the unskilled workers in the variable costs and the skilled workers in the fixed costs. This cost structure of firms does not change the qualitative insight of Krugman's original model but rather simplifies the analysis, rendering it analytically solvable (see Forslid and Ottaviano, 2003). They have dubbed their version of the CP model the Footloose Entrepreneur (FE) model[3]. It provides the means to obtain closed form solutions and express the relevant endogenous variables which are the cornerstones to explaining location decisions by firms and workers as functions of the spatial distribution of the latter.

[3]Previous versions of the model had already been put forth independently by Ottaviano (1996) and Forslid (1999).

A version of this model has allowed Baldwin *et al.* (2003) to discuss policy analysis on New Economic Geography.

Although rich in its conclusions, the CP model by Forslid and Ottaviano does not provide any theoretical insight on a three or more region model, which would be interesting for different reasons. As pointed out by Fujita *et al.* (1999), a universe considering only two regions stems from the attractiveness in dealing with manageable sized problems, although it seems implausible that the geographical relevance on economic activity worldwide can be reduced two a 2-region analysis. Extrapolation to a n-region case might be, however, too big a jump from the simpler case with two regions, though certainly not strange to many authors. A contribution concerning the former that is worth mentioning is that of Puga (1999) in his article about the region inequalities in income levels and industry location that stem from different levels of regional integration. He found that the relation between decreasing trade costs and agglomeration might be non-monotonic if the underlying integration is not accompanied by an increase in inter-regional mobility. The result is rather surprising, as even if a decrease in initially high trade costs (which entails, accordingly, an initially dispersed industry) favours agglomeration of industry, there seems to be a threshold below which, if workers are not allowed to migrate in response to wage differentials, firms will become increasingly sensitive to these differentials and will begin to disperse over time. Puga uses a more general framework and relaxes some of the assumptions found in some well known NEG models, such as those by Krugman (1991b), Krugman and Venables (1995) and Venables (1996).[4] However, in this context, what we want to highlight in the work of Puga is his generalization of the study of the dynamic properties of symmetric equilibria[5] to a n-region scenario.

Other contributions include those in the Racetrack Economy by Fujita *et al.* (1999). The authors consider a set of n regions that are equally spaced around the circumference of a circle. Transport costs are the same between each pair of adjacent regions, as they

[4] The second and third works address the problem of international barriers to labour mobility which may limit its role in explaining clustering of industries in an international context. However, as is mentioned by Puga (1999), they rely on very restrictive assumptions.

[5] Equilibrium at which all regions have identical values for all endogenous variables.

are equidistant from each other. Numerical results evidence a pattern of an equal concentration of industry in two of the n regions and, in particular, some sort of regularity in that these regions are placed at exact opposite sides of the circle. Before dealing with the n-region case in the Racetrack Economy, Fujita *et al.* (1999) also addressed the particular 3-region case. However, as in the Racetrack Economy, results concerning the existing equilibria and their dynamical properties are only numerical. On this matter, Castro *et al.* (2009) provide a number of analytical conclusions and numerical analysis regarding an extension to three regions of the CP model by Krugman (1991b) with two regions along with an insight on the n-region case.

Departing from the 2-region case, it is therefore a natural next step to address the 3-region case, but the inherent difficulties that might arise from it call for a more tractable version of the CP model. The main aim of this dissertation is to use analysis and techniques similar to the simpler CP model with two regions by Forslid and Ottaviano (2003) in order to build and study a FE model with three regions. It becomes evident that the inclusion of three regions is enough to show that the FE model still does not allow us to fully describe its dynamic properties analytically.

We begin by generalizing the FE model to three regions in section 2 and deriving a general expression for the nominal wages as explicit functions of the spatial distribution of skilled workers. In section 3, a detailed analysis on three possible outcomes concerning the dynamics of the CP model is made. We establish the stability conditions for geographic concentration and total dispersion equilibria and prove that the 3-region model favours geographic concentration of economic activity over the 2-region model. We also determine that location decisions do not depend so much on whether we consider two or three regions in the CP model when the percentage of local income spent in industrial goods is too low or the elasticity of substitution is very high when industry approaches that of perfect competition. We also establish conditions whereby an outcome of partial dispersion is never stable and show what we hope to be enough numerical evidence that it never is, whatever the parameter values. Albeit, nonlinearity of some stability conditions concerning trade costs does not allow us to fully assess this analytically. We finish section 3 by addressing the existence of bifurcations in the 3-region FE model. Section 4 makes some considerations on possible extensions of the FE model to n-regions, particularly related to agglomeration and total dispersion outcomes and section 5 is left for some concluding remarks.

2 The model

The economy is comprised of three regions that are assumed to be structurally symmetric in all aspects and equidistant from each other. All the assumptions on the economics of the model pertain to those of the 2-region model by Forslid and Ottaviano (2003) and, hence, our derivations are analogous to the latter, the differences being exactly those that result from the fact that we are considering three regions instead of two. All notation follows that of Forslid and Ottaviano (2003), except that respective to the real wages, as we shall see further ahead. Even though a good deal of our construction is a replication of Forslid and Ottaviano's, we include it here for completeness. We explicitly and clearly state the differences.

2.1 Economic environment

We have skilled and unskilled labour as production factors, of whose type each worker supplies exactly one unit inelastically, with endowments being H and L, respectively. Considering the three regions, we have $H = H_1 + H_2 + H_3$ and $L = L_1 + L_2 + L_3$. The skilled workers are assumed to be self employed entrepreneurs who can move freely between the three regions, while unskilled labour is completely immobile and considered to be evenly spread across the three regions. Hence, the endowment L is such that $L_i = \frac{L}{3}$, with $i = \{1, 2, 3\}$.

In this economy, there are two production sectors. Firstly, we have the sector of agricultural goods, which produces a homogenous good under perfect competition and constant returns, and employs unskilled labour L. Secondly, there is the industrial sector, which is in charge of producing manufactures under Dixit-Stiglitz monopolistic competition (Dixit and Stiglitz, 1977) and increasing returns, and employs the skilled and mobile workforce H.

The representative consumer has the following Cobb-Douglas utility function:

$$U_i = X_i^{\mu} A_i^{1-\mu}, \tag{2.1}$$

where A_i is the consumption of agricultural products and X_i is the consumption of the composite of all differentiated varieties of manufactures, defined by:

$$X_i = \left[\int_{s \epsilon N} d_i(s)^{\frac{\sigma}{\sigma-1}} ds \right]^{\frac{\sigma}{\sigma-1}}, \tag{2.2}$$

where $d_i(s)$ is consumption of variety s of manufactures in region i, N is the mass of varieties so that n_i is the mass of varieties produced in region i (implying that $N = n_1 + n_2 + n_3$) and $\sigma > 1$ is the constant elasticity of substitution between manufactured varieties. From standard utility maximization, $\mu \in (0,1)$ is the share of manufactured goods in expenditure. Final consumers show no preferences among the varieties, however, they would like to diversify and be able to consume different varieties, thus increasing their utility. The local income Y_i comprises skilled (w_i) and unskilled (w_i^L) nominal wages, as follows:

$$Y_i = w_i H_i + w_i^L L_i. \tag{2.3}$$

Returning to the industrial sector, product differentiation ensures that each variety is produced by one firm. In order to produce x units of variety s, a firm incurs a fixed cost equal to α units of skilled labour and a marginal cost equal to β units of unskilled labour for each unit of output produced. Herein lies the main difference between Krugman's CP model's assumptions and ours (and, *a fortiori,* the 2-region model by Forslid and Ottaviano). While we consider the skilled labour to constitute only fixed costs in the model, Krugman (1991b) qualified the mobile labour as both fixed and variable costs in his CP model. It follows that the total cost of production of a firm in location i is given by:

$$TC_i(x_i) = w_i \alpha + w_i^L \beta x_i. \tag{2.4}$$

In the agricultural sector, however, units are chosen so that one unit of output requires one unit of unskilled labour L, implying a unit production cost that is equal to the unskilled nominal wage w_i^L.

The two types of goods differ in terms of inter-regional mobility. While good A is frictionless (freely traded across the three regions), trade in good X, on the contrary, is subject to iceberg costs. In this case, for one unit of the differentiated good to reach the other region, a firm has to ship $\tau \in [1, +\infty)$ units, since the remainder $\tau - 1$ "melts" in transit. Because the three regions are equidistant from each other, the trade cost structure is such that $\tau_{ij} = 1$, if $j = i$, and $\tau_{ij} = \tau$ otherwise. Inasmuch as the existing varieties of manufactures are horizontally differentiated and trade costs are the same across regions, we can establish that a consumer in region i will be indifferent between a variety from any of the other two regions.

2.2 Short-run equilibrium

We distinguish short-run from long-run equilibrium in that the former takes the spatial distribution of skilled workers as given and allows us to study the relations between variables such as skilled workers' real wages, transport costs and share of local income spent on manufactures, whereas, in the long-run, we allow skilled workers to migrate and, hence, equilibrium corresponds to a configuration whereby skilled workers have no incentives to migrate.

In this subsection we are concerned with short-run equilibrium and obtaining expressions for real wages that will allow us to study long-run equilibria in section 3.[1] Some of the following results are analogous to Forslid and Ottaviano (2003) and are clearly indicated.

First, looking at the demand side, the representative consumer in region i maximizes utility (2.1) subject to the following budget constraint:

$$\sum_{j=1}^{3} \int_{s \epsilon N} p_{ji}(s) d_{ji}(s) ds + p_i^A A_i = Y_i, \ i = \{1, 2, 3\}, \tag{2.5}$$

where p_i^A is the price of the agricultural good, $d_{ji}(s)$ is the demand by residents in location i of a variety produced in location j and p_{ji} its price.

The optimization problem yields the following CES demand for d_{ji}:

$$d_{ji}(s) = \frac{p_{ji}(s)^{-\sigma}}{P_i^{1-\sigma}} \mu Y_i, \tag{2.6}$$

[1]Migration, as we shall see, depends entirely on indirect utility differentials as given by real wages.

with the local CES price index P_i associated with (2.2):

$$P_i = \left[\sum_{j=1}^{3} \int_{s\epsilon N} p_{ji}(s)^{1-\sigma} ds \right]^{\frac{1}{1-\sigma}} . \tag{2.7}$$

Turning now to the supply side and starting with the agricultural sector, absence of transport costs in trade of good A means that its price is the same everywhere, so $p_1^A = p_2^A = p_3^A$. Furthermore, under perfect competition, we have marginal cost pricing so that $p_i^A = w_i^L$ and we have zero profits for this sector in equilibrium. Consequently, the aforementioned marginal cost pricing rule implies wage equalization between regions ($w_1^L = w_2^L = w_3^L$). This suggests choosing A as numeraire, so that $p_i^A = w_i^L = 1$. We assume that the non-full-specialization (NFS) condition (Baldwin *et al.,* 2003) holds, guaranteeing that agriculture is active in the three regions.[2]

In the industrial sector, given the fixed cost α in (2.4), skilled labour market clearing gives us the number of firms in each region in equilibrium (same as in Forslid and Ottaviano, 2003):

$$n_i = \frac{H_i}{\alpha}.$$

A manufacturing firm in region i facing the total cost in (2.4) maximizes the following profit function:

$$\prod_i(s) = \sum_{j=1}^{3} p_{ij}(s) d_{ij}(s) - \beta \left[\sum_{j=1}^{3} \tau_{ij} d_{ij}(s) \right] - \alpha w_i, \tag{2.8}$$

where $w_i^L = 1$. Because total supply to region $j \neq i$ must include the part $1 - \tau$ of the product that melts away, it is equal to $\tau d_{ij}(s,t)$. The first order condition for maximization of (2.8) renders:

$$p_{ij}(s) = \tau_{ij} \beta \frac{\sigma}{1-\sigma}. \tag{2.9}$$

[2]The condition requires world expenditure on good A to be greater than the maximum total production of A in two regions, i.e., $(1-\mu)Y^W > \frac{2}{3}L$, where Y^W is the global income.This is guaranteed if we assume that $\mu < \sigma/(3\sigma - 2)$. In the 2-region model by Forslid and Ottaviano (2003), the corresponding assumption is $\mu < \sigma/(2\sigma - 1)$.

This pricing equation is intrinsically different from that of the original CP model in that it does not depend on the wages of skilled workers, but on the wages of unskilled workers, which are equalized across regions. It is also equivalent to that of the 2-region model by Forslid and Ottaviano. This is what renders the FE model solvable. Using (2.9), the CES price index (2.7) becomes:

$$P_i = \beta \frac{\sigma}{1-\sigma} \left[\sum_{j=1}^{3} \phi_{ij} n_i \right]^{\frac{1}{1-\sigma}}, \tag{2.10}$$

where $\phi_{ij} = \tau_{ij}^{1-\sigma} \in (0, 1]$ is what Forslid and Ottaviano have coined with the expression "freeness of trade". It increases as the transport costs τ fall, reducing the price index P_i.

Absence of entrance barriers in the manufacturing industry means that there is free entry and exit, which translates in zero profits in equilibrium. This implies that operating profits must totally compensate fixed costs in terms of skilled labour and they are equal to the wages paid to entrepreneurs (i.e., skilled workers), such that:

$$w_i = \sum_{j=1}^{3} p_{ij}(s) d_{ij}(s) - \beta \left[\sum_{j=1}^{3} \tau_{ij} d_{ij}(s) \right],$$

which becomes, considering the prices in (2.9):

$$w_i = \frac{\beta x_i}{\sigma - 1}, \tag{2.11}$$

where $x_i = \sum \tau_{ij} d_{ij}(s)$ is total production by a manufacturing firm in region i. Using (2.6), (2.9) and (2.10), we can derive an expression for x_i that depends on the local incomes and the number of firms of the three regions:

$$x_i = \frac{\mu(\sigma - 1)}{\beta\sigma} \sum_{j=1}^{3} \frac{\phi_{ij} Y_i}{\sum_{m=1}^{3} \phi_{mj} n_m}. \tag{2.12}$$

A new expression for the nominal wage can now be derived. Replacing (2.12) in (2.11) and knowing that $n_i = \frac{H_i}{\alpha}$ we have:

$$w_i = \frac{\mu}{\sigma} \sum_{j=1}^{3} \frac{\phi_{ij} Y_j}{R_j}, \tag{2.13}$$

where $R_j = \sum_{m=1}^{3} \phi_{mj} H_m$. By (2.3), income equals:

$$Y_i = \frac{L}{3} + w_i H_i. \tag{2.14}$$

The endogenous variables n_i, p_i, w_i, x_i and Y_i can be determined for a given allocation of skilled workers H. Recall that these results are analogous to those in Forslid and Ottaviano (2003), with the exception that we are considering a 3-region model. Of course, it is very straightforward to see that, starting from the 2-region model, building a n-region FE model would practically be tantamount to building a 3-region one, as we have done so far. Until this point, that is.[3] The only changes required for obtaining general expressions for n-regions would be changing the superscripts from 3 to n in the aforementioned summations and changing the amount of unskilled labour in each region to $L_i = \frac{L}{n}$.

Location decisions by skilled workers from one region to another are assumed to depend on indirect utility differentials as given by real wages. Hence, it is important to first derive the system of three linear equations in w_i, for $i = 1, 2, 3$, using (2.13) and (2.14), that can be solved to obtain the equilibrium skilled (nominal) wages as explicit functions of the spatial distribution of skilled workers H_i. This is what we do in the following:

Proposition 2.1. *The nominal skilled wages in region i are given by:*

$$w_i = \frac{\dfrac{\mu}{\sigma}\dfrac{L}{3}\left\{\displaystyle\sum_{j=1}^{3}\frac{\phi_{1j}}{R_j} + \frac{\mu}{\sigma}\left[\phi(\phi-1)\frac{\displaystyle\sum_{k\neq i} H_k}{\displaystyle\prod_{k\neq i} R_k} + \frac{\phi^2-1}{R_i}\sum_{k\neq i}\frac{H_k}{R_k}\right] + \frac{\mu^2}{\sigma^2}\left(2\phi^3 - 3\phi^2 + 1\right)\frac{1}{R_i}\prod_{k\neq i}\frac{H_k}{R_k}\right\}}{1 - \dfrac{\mu}{\sigma}\displaystyle\sum_{j=1}^{3}\frac{H_j}{R_j} + \frac{\mu^2}{\sigma^2}(1-\phi^2)\left(\frac{H_1H_2}{R_1R_2} + \frac{H_1H_3}{R_1R_3} + \frac{H_2H_3}{R_2R_3}\right) - \frac{\mu^3}{\sigma^3}(2\phi^3 - 3\phi^2 + 1)\prod_{j=1}^{3}\frac{H_j}{R_j}}. \tag{2.15}$$

Proof. See Appendix A. □

Although similar, the 2-region model cannot be derived from the present one by eliminating skilled workers from a single region.

[3]Some of the further results are already quite cumbersome with only three regions, let alone with four or more.

Remark. The 3-region model does not contain the 2-region model in Forslid and Ottaviano.

Proof. If there is no third region, the unskilled labour L has to be divided equally between two regions, hence $L_i = \frac{L}{2}$. This changes the structure of the model, as the local income Y_i is now different. We would also have to exclude $\frac{1}{R_3}$ from (2.13). Hence, simply setting the amount of skilled workers H_i to zero would, thus, not give us the expression for the nominal skilled wage in the 2-region model (see Forslid and Ottaviano, 2003). In other words, although similar, the 2-region model cannot be derived from the present one by eliminating skilled workers from a single region. $\square$

The nominal skilled wages can also be expressed as functions of the share of skilled workers in each region, $h_i = \dfrac{H_i}{H}$. Furthermore, we can conveniently omit one of the regions in our analysis, since $H_k = H - H_i - H_j$, with $i, j, k = \{1, 2, 3\}$, and define region k implicitly as function of regions i and j. This allows us to limit our analysis of the dynamics on the 2-simplex (because $h_i + h_j + h_k = 1$). Taking this into consideration, we can rewrite the numerator and denominator of (2.15), and therefore the nominal wage w_i, in terms of h_i and h_j, leaving out region h_k, as it is implicitly defined a function of h_i and h_j:

$$w_i\left(h_i, h_j\right) = \frac{Dw_i\left(h_i, h_j\right)}{D\left(h_i, h_j\right)},$$

where

$$\begin{aligned} Dw_i &= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left\{\sum_{m=1}^{3}\frac{\phi_{im}}{r_m} + \frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{h_j + h_k}{r_j r_k} + \frac{\phi^2 - 1}{r_i}\left(\frac{h_j}{r_j} + \frac{h_k}{r_k}\right)\right] + \right. \\ &\left. + \frac{\mu^2}{\sigma^2}\left(2\phi^3 - 3\phi^2 + 1\right)\frac{1}{r_i}\frac{h_j h_k}{r_j r_k}\right\}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} D &= 1 - \frac{\mu}{\sigma}\sum_{j=1}^{3}\frac{h_j}{r_j} + \frac{\mu^2}{\sigma^2}\left(1 - \phi^2\right)\left(\frac{h_1 h_2}{r_1 r_2} + \frac{h_1 h_3}{r_1 r_3} + \frac{h_2 h_3}{r_2 r_3}\right) \\ & - \frac{\mu^3}{\sigma^3}\left(2\phi^3 - 3\phi^2 + 1\right)\frac{h_1 h_2 h_3}{r_1 r_2 r_3}, \end{aligned} \tag{2.17}$$

with $r_i = \frac{R_i}{H}$. Obviously, we have $r_k = 1 + (\phi - 1)(h_i + h_j)$, since $h_k = 1 - h_i - h_j$[4]. This form of presentation is convenient for derivation purposes. Additionally, the price index P_i becomes, after (2.9):

$$P_i(h_i, h_j) = \beta \frac{\sigma}{\sigma - 1} \left(\frac{H}{\alpha} r_i \right)^{\frac{1}{1-\sigma}}. \tag{2.18}$$

[4]It does no harm to express D directly a function of h_1, h_2 and h_3, as can be understood by reading Appendix A.

3 Long-run equilibria and stability

Recall that skilled workers are assumed to base their location decisions on the difference between each region's real wage and the weighted average real wage in the three regions. This assumption follows Krugman (1991b) in that it states that the skilled workers are short sighted and migrate to the location that offers them the highest indirect utility. We define long-run equilibria[1] as distributions of skilled workers that remain unchanged over time. An equilibrium is stable if, after occurrence of some small exogenous migration of skilled workers to any of the regions, the spatial distribution of skilled workers is pulled back to the initial one. The next paragraph contains a brief description of the dynamical system of the 2-region FE model by Forslid and Ottaviano, since it is the benchmark to our 3-region model.

In the 2-region FE model, skilled worker migration follows a simple Marshallian adjustment[2], whereby the rate of change of skilled workers in one region depends on the real wage differential between the two regions adjusted by some positive real parameter. If the real wage in region i is higher than in region j, skilled workers in region j will migrate to region i. Since the shares of skilled workers in the two regions sum up to unity, the authors chose to study the dynamics through the perspective of one region only, with h being the share of skilled workers in that region. Naturally, the share of skilled workers in the other region is implicitly defined as $1-h$, so studying the dynamics in one region renders complete information on the dynamics in general. Working with two regions pertains to studying dynamics on a 1-simplex, that is, a line segment where h goes from zero to unity. The dynamics are well defined so as to capture both interior and boundary dynamics[3]. As a consequence of their formulation, interior equilibria ($h = \frac{1}{2}$) always exist, whereas corner solutions ($h = 0$ and $h = 1$) are not necessarily equilibria. However, when they are equilibria, they are necessarily stable.

Turning to our 3-region FE model, dynamics are described in a straightforward way,

[1]Hereinafter, we shall refer to long-run equilibria just as equilibria.

[2]See Forslid and Ottaviano (2003: 234-235).

[3]This refers to dynamics at points placed on the boundaries of the simplex. These correspond to $h = 0$ or $h = 1$, i.e., the boundaries of the line segment.

albeit different from that in the 2-region model. Workers migrate to a region if the wage in that region is higher than the weighted average real wage of the three regions, unless the former is initially absent of skilled workers, in which case it will be left empty unless there is some exogenous migration to that region. Choosing to leave the dynamics in region 3 implicitly defined[4], without loss of generality, the migration dynamics of skilled workers are determined by the following system:

$$\begin{cases} \dot{h}_1 = \Delta\omega_1 = & (\omega_1 - \bar{\omega})\, h_1 \\ \dot{h}_2 = \Delta\omega_2 = & (\omega_2 - \bar{\omega})\, h_2 \end{cases}, \quad h_1, h_2 \in [0,1]\,, \tag{3.1}$$

where $\omega_i = \frac{w_i}{P_i^{\mu}}$ stands for the real wage in region i and $\bar{\omega}(h_1, h_2) = h_1\omega_1 + h_2\omega_2 + h_3\omega_3$ is the weighted real wage average of wages in the three regions. The equations of the system are *ad hoc* migration equations, a standard formulation in many NEG models' dynamics, such as in the Core Periphery model in Fujita *et al.* (1999)[5] and in the version of the FE model in Baldwin *et al.* (2003) based on the original FE model by Forslid and Ottaviano (2003). It clearly shows that the rate of change of the share of skilled workers is proportional to the real wage differential. We have dynamics subject to a two-dimensional simplex (an equilateral triangle) whose boundaries correspond to configurations (h_1, h_2) such that one of the regions is left absent of skilled workers. Such configurations pertain to either $h_1 = 0$, $h_2 = 0$, or $h_1 + h_2 = 1 \Leftrightarrow h_3 = 0$.

Most criticism towards the *ad hoc* migration equations concern the fact that there is no motion when skilled workers are fully agglomerated in one region, even if there are non-null real wage differentials (Matsuyama, 1991) at such configurations, hence preventing us from explaining the lack of motion when this is the case. We should then just briefly clarify why the *ad hoc* migration equations are used in this dissertation.

It would seem as if the dynamics could be explained in a more straightforward way if we did not multiply each wage differential by the share h_i. However, we are dealing with both interior and boundary dynamics, which are more complex in a 3-region model

[4]That is, $h_3 = 1 - h_1 - h_2$.

[5]They justify its use on the grounds that their CP model might be regarded as an evolutionary game. Evolutionary game theory recurrently use "replicator dynamics" in the fashion of such equations.

than in the 2-region model. The extra h_i in each equation is necessary to guarantee that the rate of change of skilled workers in each region is null when that region is absent of skilled workers. The absence of the extra h_i would leave the dynamics ill-defined at the boundaries. For instance, imagine that both regions 2 and 3 have no skilled workers. The average real wage $\bar{\omega}$ is equal to the real wage in region 1, ω_1, so we would have a null wage differential $\Delta\omega_1$, but a differential $\Delta\omega_2$ whose signal would depend on which real wage was higher, ω_1 or ω_2. Assume, hypothetically, that the real wage is higher in region 2. It follows that $\dot{h}_1 = 0$ and $\dot{h}_2 > 0$. Furthermore, $\dot{h}_3 = -\dot{h}_2 < 0$. But symmetry between regions means that when both regions 2 and 3 are absent of skilled workers and their real wages are equal, then their rates of change should be the same, hence $\dot{h}_3 > 0$, which is a contradiction. The system would not give us enough information on the dynamics at the boundaries. On the other hand, the migration equation commonly used assures that both rates of change are equal to zero, when the regions are empty[6]. Furthermore, assuming the mentioned hypothetical dynamics, one could easily verify that the dynamic properties of a configuration such as $(1, 0, 0)$ would be different from that of $(0, 0, 1)$, so agglomerating in region 1 would be different from concentrating in region 3 as far as stability is concerned, which makes no sense since we have three regions that are identical. With the formulation implicit in the migration equation, all three regions are treated symmetrically.[7] Hence, the dynamics implicit in the migration equation in (3.1) are more suitable for our analysis. Next, we address the equilibria of (3.1).

Based on what we have argued above, we cannot say that equilibria are strictly configurations at which there are no endogenous incentives to migrate from one region to another. This is because we have no migration at corner solutions, even if there are non-null real wage differentials. Hence, we shall use the more straightforward definition of equilibria: Configurations that satisfy $\dot{h}_1, \dot{h}_2 = 0$ are all equilibria. By direct substitution, it is then easy to see that the configurations $(h_1, h_2, h_3) = (1, 0, 0)\,, \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, ,$\left(0, \frac{1}{2}, \frac{1}{2}\right)$[8] and their permutations are all equilibria of (3.1). The first one consists of

[6] This was already argued by Baldwin *et al.* (2003).

[7] We have $\dot{h}_3 = -\dot{h}_1 - \dot{h}_2 = \ldots = h_3(\omega_3 - \bar{\omega})$.

[8] Recall that, since $h_3 = 1 - h_1 - h_2$, we are still working in two coordinates. For instance, the configuration $(0, a, 1 - a)$ is equivalent to $(h_1, h_2) = (0, a)$.

full agglomeration of industry in one of the regions, leaving the other two without any skilled workers. We call this "concentration" (or full agglomeration). The second corresponds to an equilibrium for which skilled workers are equally dispersed across the three regions. This is "total dispersion". Finally, the third configuration represents an outcome whereby skilled workers are equally dispersed across two of the three regions. This is referred to as "partial dispersion".

A relevant difference between the postulated dynamics of our 3-region model and those of the 2-region model by Forslid and Ottaviano is that corner solutions are always equilibria of the system in (3.1), whereas the corresponding corner configurations may not be equilibria in the 2-region FE model.

A key ingredient in the description of the dynamics of skilled workers in NEG models is the study of stability of the following three configurations: concentration, total and partial dispersion. The equilibrium corresponding to total dispersion is fully symmetric, while the other two are partially symmetric.[9] The stability of each equilibrium is preserved by permutation so that the same stability conditions hold for concentration or partial dispersion in any of the regions.

Equilibria are stable if, due to occurrence of some exogenous migration of skilled workers to any of the regions, the spatial distribution of skilled workers is pulled back to the initial one. We have the following result concerning configurations that are placed on a boundary of the 2-simplex:

Proposition 3.1. *A necessary condition for the stability of an equilibrium such that* $h_i = 0$ *is:*

$$\omega_i < \bar{\omega}.$$

Proof. Considering an equilibrium with $h_i = 0$, if the real wage in region i is lower than the average real wage $\bar{\omega}$, then continuity of real wages in the share of skilled workers ensures that there is a point in a neighbourhood of that equilibrium where $\omega_i < \bar{\omega}$. Thus, if there was a marginal and exogenous migration of skilled workers from any

[9]We can permute either the populated or unpopulated regions but not all of the three regions.

region $j \neq i$ to region i, a negative wage differential $\Delta\omega_i$ would drive away all the skilled workers from region i back to the region of origin until $h_i = 0$. $\square$

Notice that the previous is not sufficient.

As a side note, we should mention that we shall refrain from recurring to the standard eigenvalues study of the Jacobian matrix at equilibria located on the boundaries of the 2-simplex. This is because at boundary configurations such as $(1, 0)$ or $(0, \frac{1}{2})$ boundary dynamics impose constraints on interpreting the meaning of some partial derivatives at points located on it.

3.1 Stability of concentration

As mentioned previously, the configuration $(h_1, h_2, h_3) = (1, 0, 0)$[10] is an equilibrium of (3.1). If region 2 and 3 are both absent of any skilled workers, then, by symmetry, it must hold that $\omega_2 = \omega_3$, which can be easily verified using (2.15) to compare $w_2(1, 0, 0)$ with $w_3(1, 0, 0)$ and noticing that, by (2.18), $P_2(1, 0) = P_3(1, 0)$. The equilibrium is a corner solution placed on a vertex of the 2-simplex, since we have two empty regions.

Lemma 3.2. *At the configuration* $(1, 0)$*, the necessary condition of proposition 3.1 is also sufficient.*

Proof. By proposition 3.1, a necessary condition for concentration to be a stable equilibrium is

$$\omega_2 < \bar{\omega} = h_1\omega_1 = \omega_1,$$

which is also a sufficient condition. This is because we have both $h_2 = 0$ and $h_3 = 0$ and we already know that $\omega_2 = \omega_3$. Comparing region 1 with region 2 is, thus, tantamount to comparing region 1 with region 3, so we only need the former if we are to address stability of concentration. $\square$

[10] We alternate between showing and omitting h_3 for the purpose of convenience of presentation only.

Put in economic terms, if region 2 and 3 are to remain empty over time, then there can be no incentives for skilled workers to migrate to other regions, meaning their indirect utility must be higher in region 1 compared to the other regions. This implies that skilled workers should receive a higher real wage in region 1, that is:

$$\begin{aligned} & \frac{w_1}{P_1^{\mu}} > \frac{w_2}{P_2^{\mu}} \\ \Leftrightarrow \quad & \frac{w_1}{w_2} > \tau^{\mu} \\ \Leftrightarrow \quad & \frac{w_1}{w_2} > \phi^{\frac{\mu}{1-\sigma}}, \end{aligned}$$

giving us a necessary and sufficient condition for the stability of the concentration configuration[11].

Proposition 3.3. *Full agglomeration is a stable equilibrium if and only if:*

$$SP(\phi) \equiv 1 - \frac{\mu}{\sigma} + \left(1 - \frac{\mu}{\sigma}\right)\phi + \left(1 + 2\frac{\mu}{\sigma}\right)\phi^2 - 3\phi^{\frac{\mu}{1-\sigma}+1} < 0. \tag{3.2}$$

Proof. For simplicity, we determine the condition for stability of the configuration $(h_1, h_2) = (1, 0)$ using the equivalent configuration $(H_1, H_2, H_3) = (H, 0, 0)$ and the equations in (2.13) and (2.14). This implies finding $w_1(H, 0, 0)$ and $w_2(H, 0, 0)$. After (2.15) we have:

$$\begin{aligned} w_1(H,0,0) &= \frac{\mu}{\sigma}\frac{L}{H}\frac{1}{1-\frac{\mu}{\sigma}} \\ \Leftrightarrow \quad w_1(H,0,0) &= \frac{\mu}{\sigma-\mu}\frac{L}{H}, \end{aligned}$$

[11] Even if we do not present calculations based on the study of the Jacobian, one can verify that the conditions resulting from it are equivalent to those we have obtained.

As for the nominal wage in region 2:

$$
\begin{aligned}
w_2(H,0,0) &= \frac{\mu}{\sigma}\frac{L}{3}\left[\frac{1}{\phi H}+\frac{\phi}{H}+\frac{\phi}{\phi H}+\frac{\mu}{\sigma}\left(\frac{2\phi^2-\phi-1}{\phi H}\right)\right] \\
\Leftrightarrow \quad w_2(H,0,0) &= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{\phi H}\left[\phi^2+\phi+1+\frac{\mu}{\sigma}\left(2\phi^2-\phi-1\right)\right].
\end{aligned}
$$

Substitution of nominal wages renders:

$$
\begin{aligned}
\frac{w_1}{w_2} &> \phi^{\frac{\mu}{1-\sigma}} \Leftrightarrow \\
\frac{3\phi}{\phi^2+\phi+1+\frac{\mu}{\sigma}(2\phi^2-\phi-1)} &> \phi^{\frac{\mu}{1-\sigma}} \Leftrightarrow \\
\phi^2+\phi+1+\frac{\mu}{\sigma}\left(2\phi^2-\phi-1\right) &< 3\phi^{\frac{\mu}{1-\sigma}+1} \Leftrightarrow \\
1-\frac{\mu}{\sigma}+\left(1-\frac{\mu}{\sigma}\right)\phi+\left(1+2\frac{\mu}{\sigma}\right)\phi^2-3\phi^{\frac{\mu}{1-\sigma}+1} &< 0.
\end{aligned}
$$

□

Corollary. *If* $\sigma = 1+\mu$ *then* $SP(\phi) < 0$, $\forall \phi \in (0,1]$.

Proof. This is straightforward by substitution in $SP(\phi)$. □

♠ Hereinafter, we shall assume that $\sigma > 1+\mu$.

This is the equivalent to the "no-black-hole" condition of the CP model by Krugman (1991b) and is the exact same condition found by Forslid and Ottaviano (2003) in their 2-region model. Numerous simulations suggest that, if $\sigma < 1+\mu$, concentration is also always stable[12]. The underlying economic interpretation is that, unless the elasticity of substitution σ is high enough, that is, if skilled workers do not value enough variety

[12] However, this cannot be assessed analytically. What we can doubtlessly say is that, if this condition is not verified, then the total dispersion configuration can never be a stable equilibrium. Detailed explanation on the derivation of this condition will be given in Section 3.2, where we address the stability of total dispersion.

in the composite good X, then concentration in one single region is always a stable equilibrium.

Proposition 3.4. *$SP(\phi)$ has exactly two roots in $\phi \in (0,1]$, $\phi = 1$ and $\phi_s \in (0,1)$.*

Proof. It is straightforward to see that $SP(1) = 0$ by direct substitution in (3.2), therefore $\phi = 1$ is a root of $SP(\phi)$. Also:

$$SP(0) = 1 - \frac{\mu}{\sigma} > 0.$$

Taking the first derivative of $SP(\phi)$ with respect to ϕ we obtain:

$$\frac{dSP}{d\phi} = \left(1 - \frac{\mu}{\sigma}\right) + 2\left(1 + 2\frac{\mu}{\sigma}\right)\phi - \left(\frac{\mu}{1-\sigma} + 1\right) 3\phi^{\frac{\mu}{1-\sigma}},$$

therefore, we have:

$$\frac{dSP}{d\phi}(1) = \frac{3\mu}{\sigma}\frac{1-2\sigma}{1-\sigma} > 0,$$

which, knowing that $SP(1) = 0$ and $SP(0) > 0$, ensures that there is a minimum to the left of $SP(1)$ such that $\min_\phi SP(\phi) < 0$. It also follows that there exists a zero ϕ_s of $SP(\phi)$ between 0 and 1.

Finally, to show that ϕ_s is the only zero in $(0,1)$, note that $SP(\phi)$ is convex between 0 and 1:

$$\frac{d^2SP}{d\phi^2}(\phi) = 2\left(1 + 2\frac{\mu}{\sigma}\right) - \frac{\mu}{1-\sigma}\left(\frac{\mu}{1-\sigma} + 1\right) 3\phi^{\frac{\mu}{1-\sigma}-1},$$

which is always positive since $\sigma > 1 + \mu$. Given that $SP(\phi)$ is continuous in its domain, the previous conditions ensure that there exists a zero of $SP(\phi)$, $\phi_s \in (0,1)$, and that it is unique. $\square$

The value $\phi_s < 1$ whose existence is guaranteed in proposition 3.4 is the 3-region "sustain point" (name used by Fujita *et al.,* 1999). It is the threshold of ϕ above which concentration is a stable equilibrium. It is expected that, for a high enough freeness of trade, or, conversely, for low enough transport costs, full agglomeration of industry in one of the regions is a stable outcome. This is because low transport costs means that price indices become relatively higher in the regions that are deserted and thus real wages become relatively lower.

Although we have determined a necessary and sufficient condition for stability of concentration, we were not able to derive an analytical expression for the sustain point ϕ_s. However, we are able to derive a more restrictive condition for stability of agglomeration, giving us an explicit value of ϕ that we can work with.

Proposition 3.5. *Concentration is stable if* $\phi > \phi_{ss} = \dfrac{\sigma - \mu}{\sigma + 2\mu}$.

Proof. Consider the condition derived in (3.2). We can state the following:

$$SP(\phi) < 1 - \frac{\mu}{\sigma} + \left(1 - \frac{\mu}{\sigma}\right)\phi + \left(1 + 2\frac{\mu}{\sigma}\right)\phi^2 - 3\phi,$$

since $\frac{\mu}{1-\sigma} + 1 < 1$. Rearranging the expression to the right-hand side of the inequality above, and equating it to zero, entails:

$$\begin{aligned} \sigma - \mu + (\sigma - \mu)\,\phi + (\sigma + 2\mu)\,\phi^2 - 3\phi\sigma &= 0 \Leftrightarrow \\ \Leftrightarrow \phi_{ss} &= \frac{\sigma - \mu}{\sigma + 2\mu}. \end{aligned}$$

Hence, knowing, that $\phi_{ss} > \phi_s$, we can say that concentration is stable if $\phi > \phi_{ss}$. □

Notice that this condition is sufficient only and might be too restrictive, as it requires transport costs below those that would already allow for stability of concentration.

Proposition 3.6. *As σ approaches infinity or μ approaches zero, concentration becomes an unstable equilibrium.*

Proof. We already know that full agglomeration is only stable if $SP(\phi) < 0$. However, notice the following limits:

$$\lim_{\sigma\to\infty} SP(\phi) = (\phi - 1)^2 \text{ and } \lim_{\mu\to 0} SP(\phi) = (\phi - 1)^2.$$

This means that $\phi_s \to 1$. For a given $\phi \in (0, 1)$, concentration becomes unstable for a sufficiently high σ or a sufficiently low μ. □

Proposition 3.6 handles two limit cases whereby concentration can never be a stable outcome. The first limit case ($\mu \to 0$) refers to a situation of absence of the manufacturing sector, as if μ approaches zero, there is no local income spent on manufactures. The second one corresponds to the manufacturing sector operating under perfect competition, since σ close to infinity means that variety in good X is not valued at all, so it is as if X were a homogenous good.

3.1.1 Comparing concentration in the 3-region and 2-region models

An interesting result would be to compare the 3-region model with the 2-region one in terms of stability of full agglomeration in one region. In Castro *et al.* (2009), it is proven that, in an extension of Krugman's CP model to three regions, more regions favour concentration as an outcome. Here, we obtain the following analogous result.

Proposition 3.7. *The parameter region for which concentration is stable in the* 3*-region model contains that in the* 2*-region model.*

Proof. In the model with two regions, following Forslid and Ottaviano (2003), concentration is a stable outcome for the 2-region model if:

$$1-\frac{\mu}{\sigma}+\left(1+\frac{\mu}{\sigma}\right)\phi^{2}-2\phi^{\frac{\mu}{1-\sigma}+1}\leq 0. \tag{3.3}$$

The difference between the left-hand sides (LHS) in (3.3) and (3.2) is given by:

$$D_{SP}(\phi,\mu,\sigma)=LHS(3.3)-LHS(3.2)=\frac{\mu}{\sigma}\phi(1-\phi)+\phi\left(\phi^{\frac{\mu}{1-\sigma}}-1\right). \tag{3.4}$$

By inspection of (3.4):

$$1>\phi \text{ and } \phi^{\frac{\mu}{1-\sigma}}>1,$$

thus D_{SP} is always positive. Thus, if the concentration configuration is stable in the FE model with two regions, then it is also stable in the model with three regions. □

In other words, the 3-region model favours concentration over the 2-region model. In figure 3.1 we plot the curves depicting the stability conditions of the 2-region and 3-region models. The dashed curve represents the function $SP\left(\phi\right)$ for the 2-region model and the other one corresponds to the homologous function for the 3-region model, for $\mu=0.4$ and $\sigma=5$. One can see clearly that $SP\left(\phi\right)$ becomes negative for a lower ϕ_s

in the 3-region case, thus illustrating the wider area of stability of concentration in the 3-region model compared to the model with two regions.

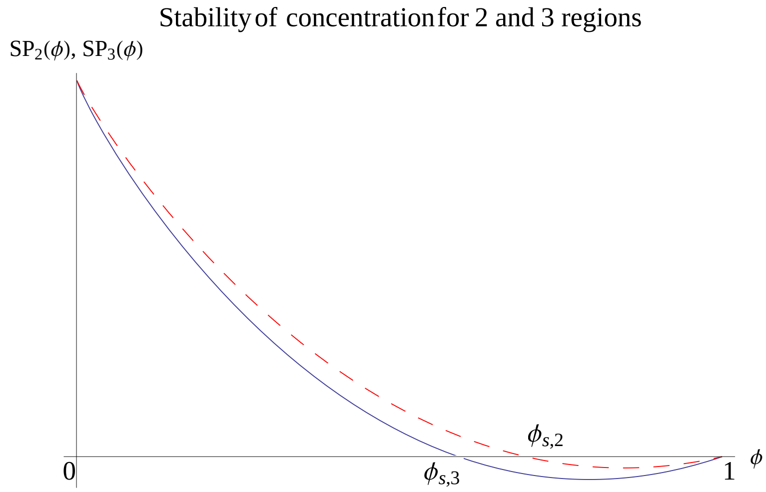

Figure 3.1 – Setting the parameters $\mu = 0.4$ and $\sigma = 5$, we plot the functions $SP_2(\phi)$ (dashed) and $SP_3(\phi)$ (solid). Since concentration is stable to the right of the zeros, we see that concentration in the 3-region model is a sufficient condition for that of the 2-region model.

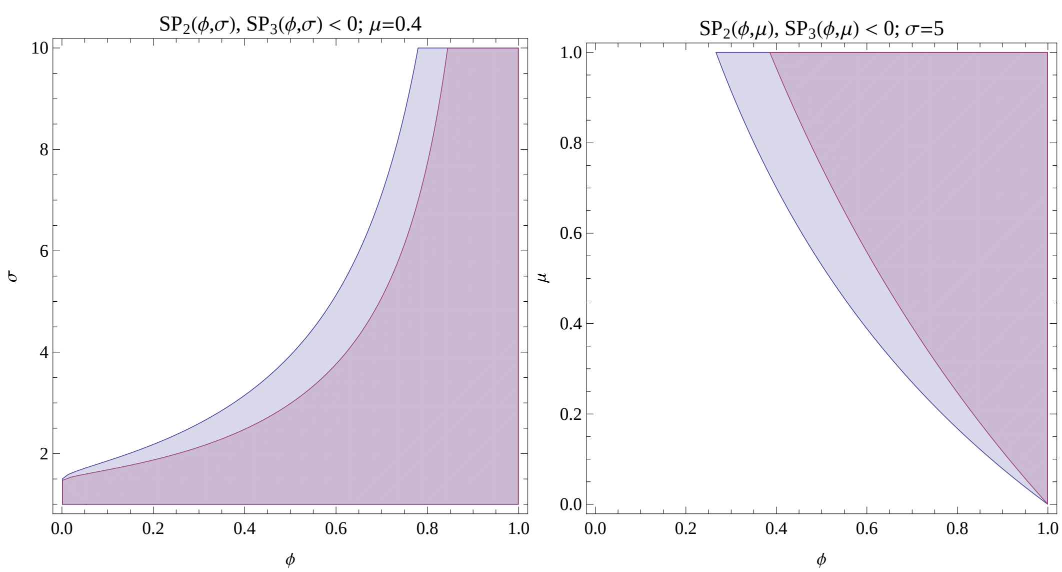

Figure 3.2 – Darker region represents region of stability for the 2-region model ($SP_2 < 0$). The lighter one represents stability for the 3-region model ($SP_3 < 0$) and clearly contains that of the 2-region model. On the left, μ is fixed and SP is plotted as a function of both ϕ and σ. On the right, σ is fixed.

Figure 3.2 illustrates the previous proposition for a wider array of values of μ and σ. The picture to the left shows the region where $SP(\phi, \sigma) < 0$, that is the region of stability of full agglomeration, for both the 2-region and 3-region models, for every ϕ and σ and fixed $\mu = 0.4$. The picture to the right has the same meaning except that we set $\sigma = 5$ and display $SP(\phi, \mu) < 0$. In both pictures it is obvious that the region where concentration is a stable outcome in the 3-region model (lighter area) contains that in the 2-region model (darker area).

Numerical results also indicate that the difference in the critical areas between the two regions gets smaller as σ increases and/or μ decreases.

Proposition 3.8. *As σ approaches infinity or μ approaches zero, the stability conditions for the concentration configuration in the 2-region and 3-region models become identical.*

Proof. If the difference D_{SP} in (3.4) equals zero, then the conditions for stability in the 2 and 3-region FE models coincide.

Indeed, we have:

$$\lim_{\mu \to 0} D_{SP} = -\phi + \phi = 0 \text{ and } \lim_{\sigma \to \infty} D_{SP} = -\phi + \phi = 0,$$

concluding the proof. □

Intuitively, and not surprisingly, what is here patent is that a smaller weight of the manufacturing sector in the whole economy reduces the relevance of the number of regions considered in that same economy, for the purpose of determining industry location. The same can be said in the case of a manufacturing sector whose variety in its good X is less valued by consumers. Note however that, by proposition 3.6, the limit cases correspond to a scenario where concentration is never stable in either the 2-region or the 3-region model.

3.2 Stability of total dispersion

Stability of an outcome where skilled workers remain equally divided across the three regions, i.e. $(h_1, h_2, h_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, is determined by the eigenvalues of the Jacobian matrix of the system in (3.1):

$$J = \begin{bmatrix} \dfrac{\partial \Delta\omega_1\left(\frac{1}{3}, \frac{1}{3}\right)}{\partial h_1} & \dfrac{\partial \Delta\omega_1\left(\frac{1}{3}, \frac{1}{3}\right)}{\partial h_2} \\ \dfrac{\partial \Delta\omega_2\left(\frac{1}{3}, \frac{1}{3}\right)}{\partial h_1} & \dfrac{\partial \Delta\omega_2\left(\frac{1}{3}, \frac{1}{3}\right)}{\partial h_2} \end{bmatrix}.$$

The region under whose perspective we are going to analyze the stability of total dispersion is a matter of choice, since the regions are all identical. Invoking this symmetry between regions, if $h_1 = h_2$ and $\omega_1(h_1, h_2) = \omega_2(h_2, h_1)$, then $\frac{\partial \omega_1}{\partial h_1}\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{\partial \omega_2}{\partial h_2}\left(\frac{1}{3}, \frac{1}{3}\right)$ and $\frac{\partial \omega_1}{\partial h_2}\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{\partial \omega_2}{\partial h_1}\left(\frac{1}{3}, \frac{1}{3}\right)$. Furthermore, it is obvious that $\frac{\partial \bar{\omega}}{\partial h_1}\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{\partial \bar{\omega}}{\partial h_2}\left(\frac{1}{3}, \frac{1}{3}\right)$. Therefore, the Jacobian J at total dispersion corresponds to:

$$J = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}.$$

Concerning the real wage average $\bar{\omega}$ we provide the following results:

Proposition 3.9. *Configurations of the form* $h_i = \frac{1-h_j}{2}$, *with* $0 < h_i < 1$, *entail* $\frac{\partial \bar{\omega}}{\partial h_i} = 0$.

Proof. See Appendix C. □

Proposition 3.10. *The weighted real wage average* $\bar{\omega}$ *attains a critical value when skilled workers are equally dispersed across regions.*

Proof. By proposition 3.9, we can conclude that both $\frac{\partial \bar{\omega}}{\partial h_1}\left(\frac{1}{3}, \frac{1}{3}\right)$ and $\frac{\partial \bar{\omega}}{\partial h_2}\left(\frac{1}{3}, \frac{1}{3}\right)$ are zero, since $h_1 = \frac{1-h_2}{2}$ and also $h_2 = \frac{1-h_1}{2}$. Because both partial derivatives are equal to zero, it means that the real wage average is at a critical value at total dispersion. □

We can now simplify α and β in the Jacobian:

$$\beta = \frac{\partial \omega_1}{\partial h_2}\left(\frac{1}{3}, \frac{1}{3}\right) \text{ and } \alpha = \frac{\partial \omega_1}{\partial h_1}\left(\frac{1}{3}, \frac{1}{3}\right).$$

Lemma 3.11. *At* $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ *we have* $\dfrac{\partial \omega_i}{\partial h_j} = 0$, $\forall i, j \neq i$.

Proof. Assume, by way of contradiction, that $\frac{\partial \omega_1}{\partial h_2}\left(\frac{1}{3}, \frac{1}{3}\right) > 0$. It must follow that, for a small $\varepsilon > 0$, $\omega_1\left(\frac{1}{3}, \frac{1}{3}+\varepsilon, \frac{1}{3}-\varepsilon\right) > \omega_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. However, the real wage in one region is invariant to the permutation of the share of skilled workers in the other two regions. Therefore, $\omega_1\left(\frac{1}{3}, \frac{1}{3}+\varepsilon, \frac{1}{3}-\varepsilon\right) = \omega_1\left(\frac{1}{3}, \frac{1}{3}-\varepsilon, \frac{1}{3}+\varepsilon\right)$. But saying that $\omega_1\left(\frac{1}{3}, \frac{1}{3}-\varepsilon, \frac{1}{3}+\varepsilon\right) > \omega_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is a contradiction. Hence, $\frac{\partial \omega_1}{\partial h_2}\left(\frac{1}{3}, \frac{1}{3}\right) = 0$. Symmetry establishes an analogous result for $\frac{\partial \omega_2}{\partial h_1}$. □

On account of the previous propositions and lemma, we are able to rewrite the Jacobian matrix at $\left(\frac{1}{3}, \frac{1}{3}\right)$ as:

$$J = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}.$$

The matrix has a double eigenvalue equal to α and total dispersion is equivalent to:

$$\alpha = \frac{\partial \omega_1}{\partial h_1}\left(\tfrac{1}{3}, \tfrac{1}{3}\right) < 0. \tag{3.5}$$

Stability of total dispersion depends only on whether a small increase in the share of skilled workers in region i leads to a decrease in the real wage in that same region. In accordance with the definition of stable equilibria, as explained in the beginning of this section, the previous statement is tantamount to saying that total dispersion can only be stable if, after an exogenous migration to one of the regions, say region 2, the real wage falls under the average real wage level, thus inducing skilled workers to migrate back to the regions they left ($\dot{h}_1 < 0$) and restore the initial distribution. This is exactly what the condition in (3.5) stands for. Of course, since total dispersion is a fully symmetric equilibrium, this enables us to generalize the latter result considering any region. Thus, we need only to study what happens when there is exogenous migration to one of the regions, as it is the same as if it happened to any of the other two regions.

The condition for stability of total dispersion is tantamount to:

$$\begin{array}{rcl} \dfrac{\partial \omega_1}{\partial h_1}\left(\frac{1}{3},\frac{1}{3}\right) < 0 & \Leftrightarrow & \dfrac{\partial w_1}{\partial h_i}P_1^{\mu} - \dfrac{\partial P_1^{\mu}}{\partial h_i}w_1 < 0 \Leftrightarrow \\ \dfrac{\dfrac{\partial w_1}{\partial h_1}}{w_1} & < & \dfrac{\dfrac{\partial P_1^{\mu}}{\partial h_1}}{P_1^{\mu}}. \end{array}$$

This inequality enables us to relate stability of dispersion in terms of semi-elasticities. Skilled workers remain equally dispersed across the three regions if an increase in the percentage of skilled workers in a region induces a percentage change in the nominal wage smaller than the corresponding percentage change in the real prices. In other words, a loss in real purchase power due to an increase of the share of skilled workers h_i in a region leads to an exodus of some skilled workers from that region until the initial share of skilled workers is restored, that is, until $h_i = \frac{1}{3}$.[13]

Proposition 3.12. *Total dispersion is a stable equilibrium if and only if:*

$$BP(\phi) = \mu^2(\phi - 1) + (\sigma - 1)\sigma(\phi - 1) + \mu(-1 + 2\sigma)(1 + 2\phi) < 0. \qquad (3.6)$$

Proof. See Appendix C.1. □

Here, the critical value ϕ_b such that $BP(\phi_b) = 0$ is called the "break point", following the terminology of Fujita *et al.* (1999):

$$\begin{array}{rcl} BP(\phi_b) = 0 & \Leftrightarrow & \phi_b = \dfrac{\mu + \mu^2 - \sigma - 2\mu\sigma + \sigma^2}{-2\mu + \mu^2 - \sigma + 4\mu\sigma + \sigma^2} \Leftrightarrow \\ \Leftrightarrow \phi_b & = & \dfrac{(\mu - \sigma)(1 + \mu - \sigma)}{\mu^2 + (-1 + \sigma)\sigma + \mu(-2 + 4\sigma)}. \end{array} \qquad (3.7)$$

It is also the only zero since (3.6) is clearly linear in ϕ.

Stability of total dispersion happens for values of $BP(\phi)$ such that $\phi < \phi_b$. That said,

[13] Dividing both sides of the inequalities by h_1 renders a similar relation in terms of elasticities. In the present case, however, semi-elasticities fit better with intuition, insofar as h_1 is the percentage of skilled workers in region 1.

if ϕ_b is negative, total dispersion is never a stable outcome. Like Forslid and Ottaviano (2003), we rule out this possibility by finding the threshold value of σ such that:

$$\begin{aligned} \phi_b &\geq 0 \Leftrightarrow \\ (\mu-\sigma)(1+\mu-\sigma) &\geq 0 \Leftrightarrow \\ \sigma &\geq 1+\mu. \end{aligned}$$

Hence, the "no-black-hole" condition ($\sigma > 1+\mu$) implies that ϕ_b is positive. A high ϕ, corresponding to low transport costs, naturally discourages dispersion in favour of concentration. It seems reasonable to claim that a higher percentage of local income spent on manufactures and a higher preference for variety of manufactures also favour total dispersion of skilled workers.

Proposition 3.13. *As σ approaches infinity or μ approaches zero, total dispersion is a stable outcome for all values of ϕ.*

Proof. Considering the value ϕ_b in (3.7) we have the following limits:

$$\lim_{\sigma\to\infty}\phi_b(\mu,\sigma) = \lim_{\sigma\to\infty}\frac{(\mu-\sigma)(1+\mu-\sigma)}{\mu^2+(-1+\sigma)\sigma+\mu(-2+4\sigma)} = \lim_{\sigma\to\infty}\frac{-1-2\mu+2\sigma}{-1+4\mu+2\sigma} = 1$$

and

$$\lim_{\mu\to 0}\phi_b(\mu,\sigma) = \lim_{\mu\to 0}\frac{(\mu-\sigma)(1+\mu-\sigma)}{\mu^2+(-1+\sigma)\sigma+\mu(-2+4\sigma)} = \frac{\sigma(\sigma-1)}{(-1+\sigma)\sigma} = 1.$$

Since total dispersion is a stable outcome for $\phi < \phi_b = 1$ and $\phi \in (0,1)$, we can say that total dispersion is always a stable outcome as μ approaches zero and σ approaches infinity. □

3.2.1 Comparing total dispersion in the 3-region and 2-region models

We compared stability conditions of full agglomeration outcomes in the 2 and 3-region models. These, although suggestive, do not extend to a comparison for the stability of total dispersion. The following propositions in this section allow us to fill this gap.

Proposition 3.14. *The parameter region for which concentration is stable in the* 2-*region model contains that of the* 3-*region model.*

Proof. In the article by Forslid and Ottaviano (2003), the authors determined the break-point value $\phi_{b,2}$ for the model with two regions which is presented as follows:

$$\phi_{b,2} = \phi_w \frac{\sigma - 1 - \mu}{\sigma - 1 + \mu},$$

where

$$\phi_w \in (0,1) = \frac{\sigma - \mu}{\sigma + \mu}$$

is a threshold value of ϕ above (below) which the region with more skilled workers offers a higher (lower) skilled worker wage w_i. Subtracting the 3-region break-point in (3.7) to this one yields:

$$\begin{aligned} D_{BP} &= \phi_{b,2} - \phi_b = \frac{(\mu-\sigma)(1+\mu-\sigma)}{(-1+\mu+\sigma)(\mu+\sigma)} - \frac{(\mu-\sigma)(1+\mu-\sigma)}{\mu^2 + (-1+\sigma)\sigma + \mu(-2+4\sigma)} \Leftrightarrow \\ \Leftrightarrow D_{BP} &= \frac{\mu(\mu-\sigma)(1+\mu-\sigma)(-1+2\sigma)}{(-1+\mu+\sigma)(\mu+\sigma)\left(\mu^2+(-1+\sigma)\sigma+\mu(-2+4\sigma)\right)}. \end{aligned}$$

The denominator is clearly positive and, provided that the "no-black-hole" condition holds, the numerator is also positive. Hence, $D_{BP} > 0$, $\forall \mu, \sigma$. This means that the critical value ϕ_b is lower in the 3-region model compared to the 2-region model. Therefore, dispersion is a more likely outcome in the model with two regions. □

Not surprisingly, this result is the opposite to the one proved for the concentration configuration. Considering both results, we can now conclude that the 3-region model indeed favours concentration over dispersion when compared to the 2-region model by Forslid and Ottaviano (2003).

Lastly, we have the following proposition concerning, yet again, the limit cases of a non-existent manufacturing industry and when the latter is very close to perfect competition.

Proposition 3.15. *As μ approaches zero or as σ approaches infinity, the stability conditions for the 2 and 3-region FE models coincide.*

Proof. We have the following limits:

$$\lim_{\mu \to 0} D_{BP} = \frac{0}{[(-1+\sigma)\sigma]^2} = 0 \text{ and } \lim_{\sigma \to \infty} D_{BP} = 0\ ^{14}.$$

Inasmuch as the difference between the break-points in the two models is zero considering the limits above, the conditions for stability of total dispersion become the same. □

Taking propositions 3.13 and 3.15 together, we can conclude that, considering the FE model with two or three regions, total dispersion is always a stable outcome when we are either approaching an economy absent of industry or consumers give almost no value to variety in good X. This fits well with intuition.

3.3 Simultaneity of concentration and total dispersion

Numerical inspection of the conditions of concentration and total dispersion in (3.2) and (3.6) suggests that, for every pair (μ, σ), there exists $\phi \in (0,1)$, for which both total dispersion and concentration are stable equilibria. In order to illustrate this in a clear way, we first set $\mu = 0.4$ and display the two dimensional regions $SP(\phi, \sigma) < 0$ and $BP(\phi, \sigma) < 0$ in one picture. Next, we plot $SP(\phi, \mu) < 0$ and $BP(\phi, \mu) < 0$ by setting $\sigma = 0.5$, in the same fashion as in the previous section. Figure 3.3 illustrates this situation. The intersection between the two regions is clear in both pictures and enables us to conclude that there is indeed a region in parameter space where both full agglomeration and total dispersion are stable outcomes.

[14] Notice that the numerator is a polynomial function of σ of degree 3 and the denominator is a polynomial function of σ of degree 4 and, therefore, the limit equals zero.

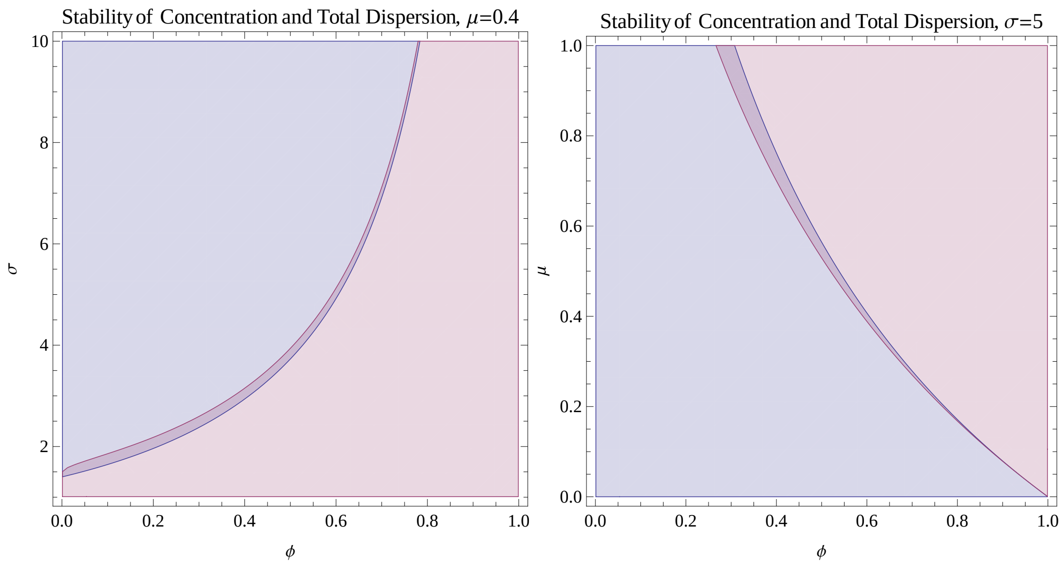

Figure 3.3 – Simultaneity of stability of concentration and total dispersion. One region (bottom left on the picture to the left and upper right on the picture to the right) corresponds to stability of concentration and the other corresponds to stability of total dispersion. They overlap each other in the darker region in the middle, where both equilibria are stable outcomes.

If this is true, then it must hold that $\phi_s < \phi_b$, because concentration is only stable for $\phi > \phi_s$, while the region for stability of total dispersion implies $\phi < \phi_b$. If it does hold, then we have hysteresis in location (Forslid and Ottaviano, 2003), because transport costs have to rise above the corresponding break point in order for total dispersion to be unstable, even if concentration is already a stable equilibrium. If both equilibria are to be simultaneously stable, we must have $SP(\phi_b) < 0$, $\forall \mu, \sigma$. Again, this seems to be the case, however, nonlinearity of SP in ϕ makes it impossible to prove this analytically. Further inspection also seems to suggest that the distance between ϕ_s and ϕ_b is bigger for parameter values near the "no-black-hole" condition.

Proposition 3.16. *There is an open subset in parameter space (ϕ, σ, μ) in which both concentration and total dispersion are stable outcomes.*

Proof. Consider the point in the parameter space $(\phi, \sigma, \mu) = \left(\frac{3}{5}, 5, \frac{2}{5}\right)$.[15]

[15]This point was chosen by numerical inspection.

At this point, we have:

$$SP\left(\tfrac{3}{5}, 5, \tfrac{2}{5}\right) < 0 \text{ and } BP\left(\tfrac{3}{5}, 5, \tfrac{2}{5}\right) < 0.$$

Therefore, for $(\phi, \sigma, \mu) = \left(\frac{3}{5}, 5, \frac{2}{5}\right)$, both concentration and total dispersion are stable equilibria. Since SP and BP are continuous functions of (ϕ, σ, μ), we know the signs persist in a open neighbourhood of $\left(\frac{3}{5}, 5, \frac{2}{5}\right)$. $\square$

We know for sure that concentration and total dispersion can be stable outcomes at the same time, even though we cannot define analytically the region where this happens. This also holds in the case of the 2-region model by Forslid and Ottaviano (2003). Figure 3.4 seems to suggest that it is always possible to find a value of ϕ, for any pair of μ and σ, such that both agglomeration and total dispersion are stable outcomes. One can see the surfaces corresponding to $SP(\phi) = 0$ and $BP(\phi) = 0$. In between the surfaces, there is simultaneity of stability and total dispersion, which is less likely for higher values of ϕ.

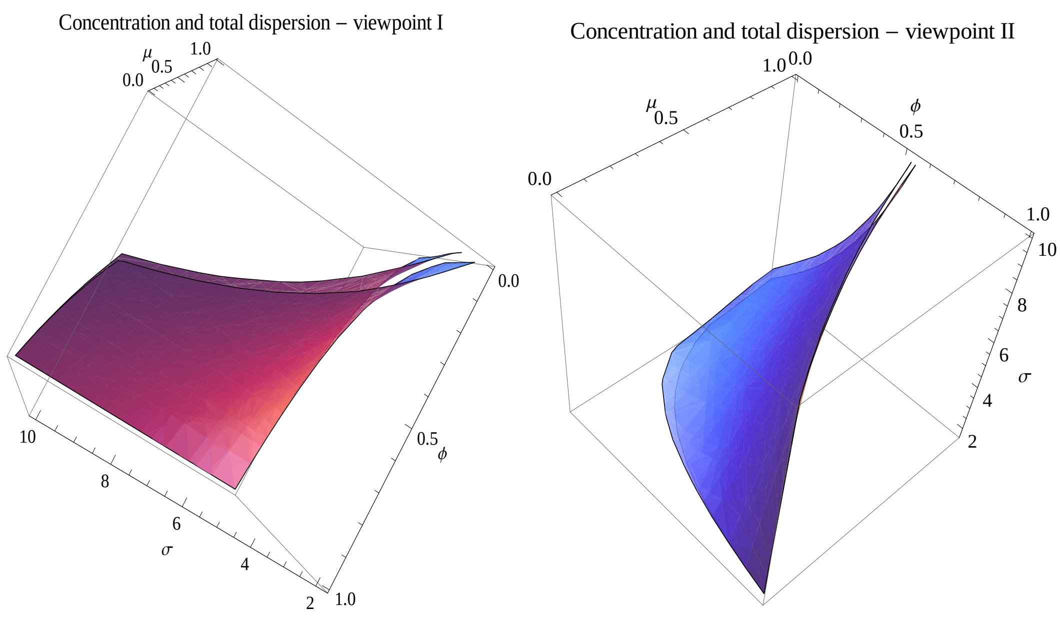

Figure 3.4 − We have the surfaces $SP = 0$ and $BP = 0$ in the parameter space (resp. top and bottom surfaces). Concentration and total dispersion are both stable in between both surfaces, where we have $SP < 0 \cap BP < 0$.

However, the region between the surfaces is very thin, and becomes thinner for a high

σ and/or low μ, thus making it harder to visualize simultaneity of concentration and total dispersion.

On a final note, consider again the limit cases, with σ tending to infinity and μ to zero, along with propositions 3.13 and 3.6. Since these pertain to cases where both the sustain point ϕ_s and the break point ϕ_b converge to unity (the corresponding transport costs converging to zero), there is no simultaneity of stability of equilibria. In fact, the aforementioned cases rule out the possibility of full agglomeration in detriment of total dispersion (which will be always stable). All these results also pertain to the 2-region model by Forslid and Ottaviano (2003), as given by propositions 3.8 and 3.15. This complements their results concerning convergence between the sustain point and break point when the manufacturing sector approaches perfect competition.

3.4 Stability of partial dispersion

The last relevant equilibrium is that for which all skilled workers are equally dispersed across two of the three regions, leaving the remaining region empty. Qualitatively, this configuration differs from the configuration $\left(\frac{1}{2}, \frac{1}{2}\right)$ in the 2-region model, meaning that the model with three regions is intrinsically different (see Castro *et al.*, 2009). Using the freedom of choice provided by symmetry, we look at the point $(h_1, h_2, h_3) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$.

Since $h_1 = 0$, we know, by proposition 3.1, that if partial dispersion is to be stable, we must have:

$$\omega_1 < \omega_2.$$

On the other hand, unlike in the case of concentration, this is not a sufficient condition. This condition states that, when $\left(0, \frac{1}{2}\right)$ is indeed an equilibrium, after an exogenous (and marginal) increase in h_1[16], the latter will eventually drop back to zero, since skilled workers expect to receive a higher real wage in any of the other regions.[17] Thus, this condition asserts that, after a deviation from the boundary of the simplex corresponding

[16] We can only consider positive variations in h_1 as we are on the boundary respective to $h_1 = 0$.
[17] Recall that ω_2 and ω_3 are equal at $\left(0, \frac{1}{2}\right)$.

to $h_1 = 0$, the spatial distribution will return to a point on that same boundary.

Yet, this condition is not sufficient because we are only comparing region 1 with regions 2 and 3. We need to ensure not only that h_1 will remain zero but also that $h_2 = h_3$, i.e., $h_2 = \frac{1}{2}$, in order to ensure convergence along the boundary (to the equilibrium). In economic terms, if any of the skilled workers migrates, e.g. to region 2, we need them to want to return to region 3 (symmetry implies the same in the opposite direction). This is achieved when an increase in h_2 leads to a decrease in the difference between the real wage ω_2 and the real wage average $\bar{\omega}$:

$$\frac{\partial \Delta\omega_2}{\partial h_2} = \frac{\partial \omega_2}{\partial h_2}\left(0, \tfrac{1}{2}\right) - \frac{\partial \bar{\omega}}{\partial h_2}\left(0, \tfrac{1}{2}\right) < 0.$$

However, since we have $h_2 = \frac{1-h_1}{2}$ at $\left(0, \frac{1}{2}\right)$, proposition 3.9 asserts that $\frac{\partial \bar{\omega}}{\partial h_2}\left(0, \frac{1}{2}\right) = 0$. Again, as in total dispersion, intuition tells us that we need only consider what happens to the real wage (considering only directions along the boundary respective to $h_1 = 0$, that is). This is because at partial dispersion the real wage ω_2 and $\bar{\omega}$ are equal. So, should ω_2 fall after an increase in h_2, this will be enough to drive skilled workers out of region 2 back to where the real wage is now relatively higher.

Proposition 3.17. *Thus, we have the following necessary and sufficient conditions for stability of partial dispersion*[18]*:*

$$\begin{cases} \xi \equiv \omega_1 - \omega_2 & < 0 \\ \beta \equiv \dfrac{\partial \omega_2}{\partial h_2}\left(0, \tfrac{1}{2}\right) & < 0. \end{cases}$$

The configuration of partial dispersion $\left(0, \frac{1}{2}, \frac{1}{2}\right)$ *is stable if and only if:*

$$\begin{cases} \xi < 0 \Leftrightarrow \sigma + \mu(-1+\phi)(1+2\phi) + \sigma\phi\left[1 + 4\phi - 3\left(2\frac{\phi}{1+\phi}\right)^{\frac{\mu}{1-\sigma}}(1+\phi)\right] & < 0 \\ \beta < 0 \Leftrightarrow 3\mu^2(-1+\phi) + 2(-1+\sigma)\sigma(-1+\phi) + \mu(-2 - 4\phi + \sigma(5+7\phi)) & < 0. \end{cases} \tag{3.8}$$

Proof. See Appendix C.2. □

[18] Once again, it should be said that these conditions are exactly the same as if we studied the Jacobian matrix at partial dispersion.

Consider the condition equivalent to $\beta < 0$. If we let ϕ_p be the single zero of the function $\beta(\phi)$ [19], then we can establish the following proposition:

Proposition 3.18. *Partial dispersion is unstable if:*

$$\phi_p < \phi < 1,$$

where:

$$\phi_p = \frac{(2 + 3\mu - 2\sigma)(\mu - \sigma)}{3\mu^2 + 2(-1 + \sigma)\sigma + \mu(-4 + 7\sigma)}.$$

Proof. If β is positive then we have a sufficient condition for the partial dispersion to be unstable. It can easily be verified that $\beta(\phi_p) = 0$. Since the expression for β is clearly linear in ϕ and an increasing function of ϕ, then partial dispersion is unstable for a ϕ above a threshold value ϕ_p. □

The previous proposition gives us a sufficient condition for the instability of partial dispersion. However, we suspect that this configuration is unstable, for all parameter values. Numerical inspection of both conditions in (3.8) suggests that these are never simultaneously met. Figure 3.5 lets us see the whole picture for fixed values of either σ or μ. We choose, as usual, $\mu = 0.4$, and plot the region of stability of partial dispersion. The upper region corresponds to $\beta < 0$ and the lower region to $\xi < 0$. That they are never simultaneously negative is straightforward to see. We can argue that partial stability will certainly never be a stable outcome if $\xi(\phi_p) > 0$. However, the difficulty we faced when comparing full agglomeration and total dispersion is present here too, because of the non-linearity in ϕ due to the exponent $\frac{\mu}{1-\sigma}$. But figure 4 seems to suggest something more. Notice that, for a value of ϕ very close to zero, the eigenvalue β is only negative above a threshold value of σ that seems to greater than the "no-black-hole" condition:

$$\beta(\phi = 0) = -3\mu^2 - 2(-1 + \sigma)\sigma + \mu(-2 + 5\sigma).$$

[19] The function $\beta(\phi)$ is clearly linear in ϕ.

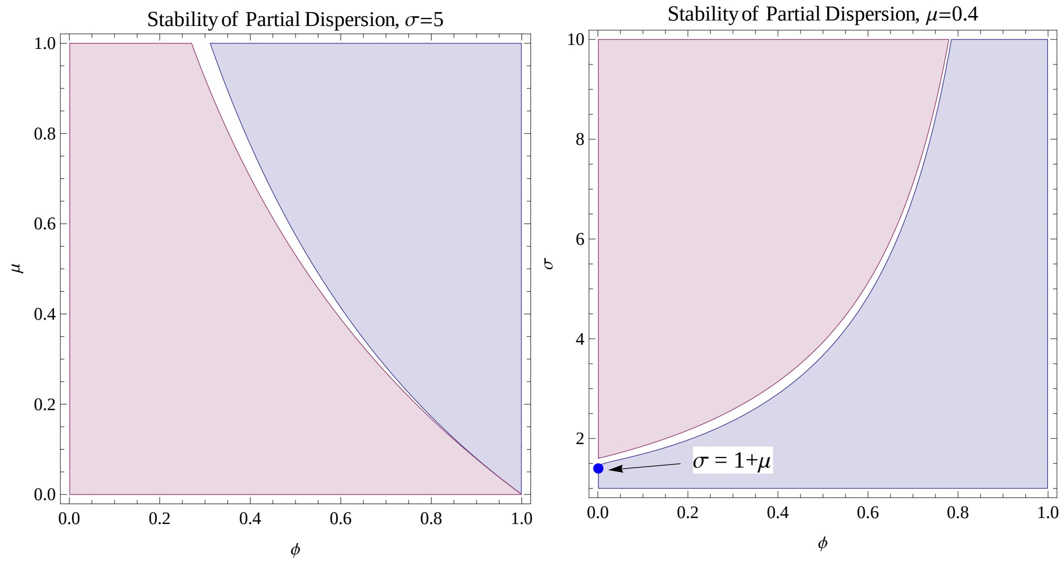

Figure 3.5 – The bottom left area in the picture to the left (resp. upper left area in the picture to the right) is the region where$\beta < 0$, while the other one corresponds to $\xi < 0$. Apparently, the conditions for stability of partial dispersion are never simultaneously met.

The threshold value of σ we are looking for is that for which we have:

$$\begin{aligned} -3\mu^2 - 2(-1+\sigma)\sigma + \mu(-2+5\sigma) &= 0 \Leftrightarrow \\ \sigma &= 1 + \frac{3}{2}\mu. \end{aligned}$$

We establish the following proposition:

Proposition 3.19. *Partial dispersion is always unstable if* $1 + \mu < \sigma \leq 1 + \frac{3}{2}\mu$.

Proof. If at least one of the eigenvalues is positive for a σ such that $1+\mu < \sigma \leq 1+\frac{3}{2}\mu$, then partial dispersion is unstable. Consider the eigenvalue β. Substitution of the upper bound $\sigma = 1 + \frac{3}{2}\mu$ entails:

$$\beta = 6\mu(1 + 3\mu)\phi > 0.$$

On the other hand, substituting the lower bound $\sigma = 1 + \mu$ renders:

$$\beta = \mu(1 + (5 + 12\mu)\phi) > 0.$$

The second derivative of β with respect to σ gives us:

$$\frac{\partial^2 \beta}{\partial \sigma^2} = 4(\phi - 1) < 0,$$

which proves that β is concave. Since β is a continuous function in σ in all its domain, the fact that it is positive at both the upper and lower limit imposed on σ in proposition 3.19 necessarily implies that it is positive in between them. Hence, the eigenvalue β is positive for $1 + \mu < \sigma \leq 1 + \frac{3}{2}\mu$ and thus partial dispersion is unstable in that interval. $\square$

It seems highly plausible that the elasticity of substitution should be above the upper bound set in proposition 3.19, since that bound is close enough to the "no-black-hole" condition. So, in order to rule out the possibility of stability of partial dispersion, we need to consider values of σ above $1 + \frac{3}{2}\mu$. Again, this appears to be very hard or even impossible to prove analytically. Representing the regions in (3.8) in the parameter space is a generalization of what we portrayed in figure 4 for every μ. Figure 3.6 contains the regions and shows clearer evidence that an equilibria where skilled workers are equally dispersed across two of the regions can never be a stable one.

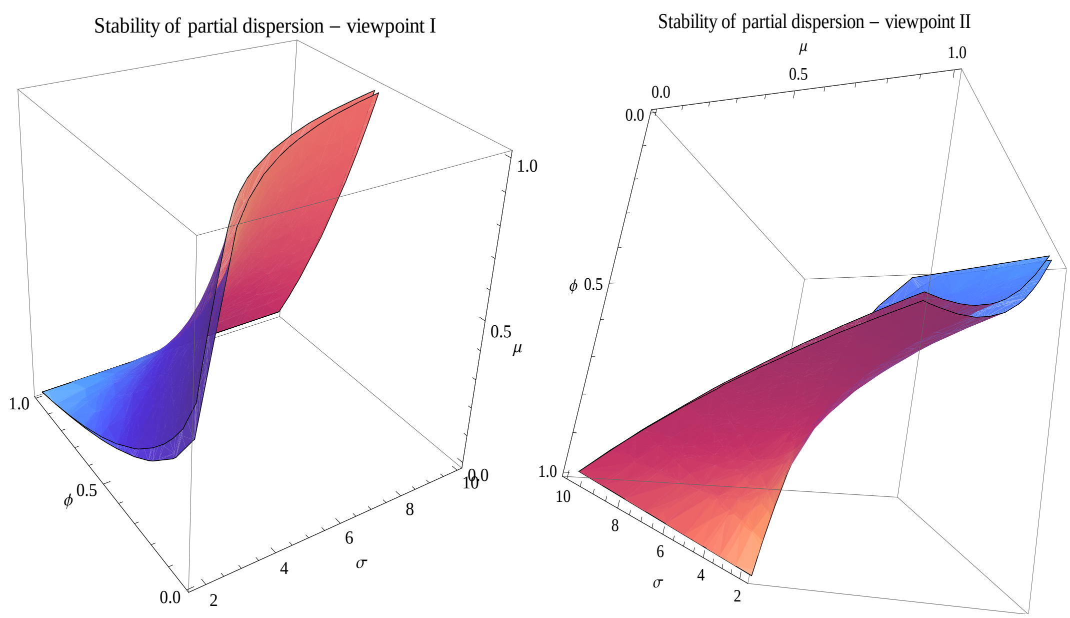

Figure 3.6 – Depiction of the surfaces $\xi = 0$ and $\beta = 0$ in parameter space (resp. top and bottom surface). Under the bottom surface we have $\beta < 0$ and above the top surface we have $\xi < 0$, hence the region between the two surfaces means that ξ and β can never be simultaneously negative.

3.5 Bifurcation in the 3-region FE model

One important feature of symmetric 2-region CP models is the existence of a "subcritical pitchfork" bifurcation[20]. It is used to explain the change in the spatial distribution of industry, for a given set of parameters, as the transport costs change. When transport costs are very high, only a symmetric equilibrium[21] is a stable outcome. However, as soon as the transport costs fall below a threshold level, two other stable equilibria will arise, those corresponding to concentration outcomes. If transport costs are low enough, the symmetric equilibrium will become unstable and full agglomeration will be the only stable outcome.

The results we have obtained so far in this section allow us to conclude that this kind of bifurcation also exists in the 3-region FE model. Instead of building a three dimensional bifurcation diagram, we adopt a similar approach to Fujita *et al.* (1999, chap. 6), where the dynamics of an extension of the CP model to three regions are portrayed inside the 2-simplex. In the previous sections, we have obtained (hopefully) enough analytical and numerical results to enable us to reduce the dynamics of the model to three distinct cases of transport costs.

All settings of parameters used in this subsection are chosen by numerical inspection. For the three simulations made, we set $\sigma = 5$ and $\mu = 0.4$. Figure 3.7 depicts the dynamics portrayed in (3.1) inside the 2-simplex for three different values of ϕ. On the picture to the left we have high transport costs ($\phi = 0.5$) and we can see that total dispersion is the only stable equilibrium, as the vector field exhibits convergence to the middle point of the simplex. The picture in the middle corresponds to a moderate level of transport costs ($\phi = 0.6$), whereby both concentration and total dispersion are stable equilibria. This means that we have $0.5 < \phi_s < 0.6$, because we know that concentration can only be stable when ϕ is above the sustain point. One can also see that the decrease in the transport costs gave rise to three new unstable equilibria between total dispersion and the concentration configurations. These were not observed

[20] This is also known as "tomahawk" bifurcation, though the latter designation might be more suitable to name the corresponding bifurcation diagram, rather than the bifurcation itself.

[21] This would be equivalent to total dispersion in our model.

in our analysis as the expressions are too complicated. Finally, when transport costs are low enough ($\phi = 0.9$), total dispersion is no longer stable and the only possible outcome is that of full agglomeration of skilled workers in one of the three regions. Here, we have $\phi_b < \phi = 0.9$. This latter case is shown in the picture to the right, where we can see convergence to either vertices of the simplex.

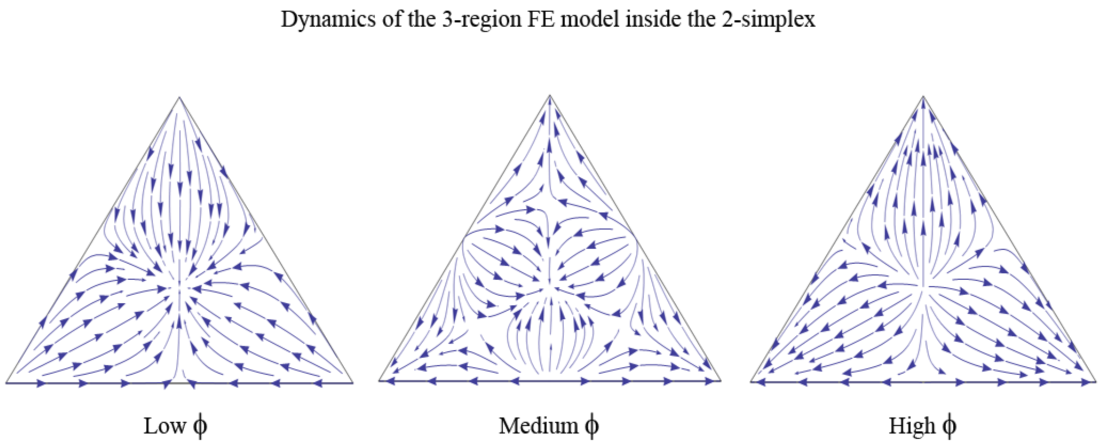

Figure 3.7 – The dynamics of the FE 3-region model. The pictures from the left to the right depict the change in the stability of equilibria as transport costs fall.

Of course, we have $\phi_s < \phi_b$, as the results obtained in section 3.3 would suggest. Another important aspect is that, throughout the three cases described in figure 3.7, partial dispersion is always unstable, which is also not surprising. All these results corroborate those in Fujita *et al* (1999, chap. 6).

In a general way, "subcritical pitchfork" bifurcations arise in most CP models where the regions concerned are fully symmetric. It is widely acknowledged that these bifurcations disappear when there are exogenous regional differences (e.g. see Baldwin *et al.*, 2003). In particular, the FE model for two regions is no exception, as is shown by Forslid and Ottaviano (2003). However, discussions remain on whether the core-periphery pattern implicit in such bifurcations is a sustainable one, even when the model on which it is based is symmetric.

In the article concerning bifurcations in migration dynamics by Berliant and Kung (2009), the authors criticize this view, on the grounds that the existence of bifurcations in symmetric CP models rest on a strategic parametrization of the model. While many authors claim that it is exogenous asymmetries alone that break core-periphery patterns based on bifurcations, Berliant and Kung argue that, while this might be one of the reasons, variations in parameters that preserve the symmetry between regions in the models may also contribute to this. Therefore, bifurcations in CP models suffer from a lack of robustness.

The parametrization used in this subsection is indeed a strategic choice to facilitate the explanation of bifurcations in the 3-region FE model. However, given the numerical evidence presented in section 3.3, it seems plausible that the dynamics implicit in the model will always undergo a “subcritical pitchfork” bifurcation, as there seems to be a region in parameter space, for $\phi \in (0,1)$, where both concentration and total dispersion are stable. Nevertheless, we have shown that this region is very thin, and hardly noticeable when σ and μ are very high. Of course, considering propositions 3.13 and 3.6 yet again, there are no bifurcations when the industrial sector faces perfect competition or when it is non-existent.

4 Note on possible extensions of the FE model to n-regions

All the results we have obtained so far pertain to an analysis concerning an extension of the FE model by Forslid and Ottaviano (2003) to three regions. Even with an analytically solvable version of the original CP model by Krugman (1991b), expressions such as the real wages as explicit functions of the spatial distribution of skilled workers seem hardly intelligible and most derivations concerning the study of dynamics seem to imply cumbersome calculations. If nothing else, one could at least imagine that further augmenting the dimension by increasing the number of regions considered in the analysis might transform it into an unmanageable problem. In particular, some issues that are impossible to address analytically in the model with three regions are likely to remain unaddressable in a model with four or more regions. Nonetheless, expanding the study of Economic Geography to include more regions should not be overlooked if the objective is to explain geographical concentration of industry in a world comprised of many inter-related regions.

Though an extension to the n-region case is not the object of this dissertation, the analysis of the 3-region model and its comparisons with the 2-region one might give some insights and serve as a motivation in the sense of extending the assumptions implicit in the FE model to a broader context. First of all, we have proved that concentration is more likely in the 3-region model (while total dispersion is less likely) in comparison with the 2-region model. This could mean that increasing the number of regions in the analysis would result in increasing the possibility of concentration (and the opposite respective to total dispersion). Castro *et al.* (2012), for instance, proved that dispersion is less likely in a 4-region model compared to the 3-region model (considering Krugman's original CP model). Another extrapolation, which may seem very straightforward, is that of considering the limit cases of μ approaching zero and σ approaching infinity on a n-region model. It would seem reasonable to claim that, in both these cases, dispersion is always stable (while concentration never is). Further results would most likely require a formal application of the FE model to a n-region context. A good next step would be to consider n regions equally spaced around a circumference, with equal trade costs between each adjacent region, and build the FE model around these assumptions. Perhaps the simplifying and uncompromising assumptions implicit in the

FE model by Forslid and Ottaviano (2003) will allow for a complement on the results of the Racetrack Economy (Fujita *et al.*, 1999), which provides the aforementioned set-up.

5 Conclusion

Building on the 2-region FE model by Forslid and Ottaviano (2003), we have obtained both analytical and numerical results from a FE model with three regions, which we have constructed along the lines of Forslid and Ottaviano. These results corroborate those already obtained in previous works on 3-region Core Periphery models. We have shown that the 3-region FE model favours concentration in comparison with the 2-region one. Furthermore, we have proved analytically that both concentration and full dispersion can be simultaneously stable and provided numerical evidence in that, for every pair (μ, σ), there exists $\phi \in (0, 1)$ where this is possible, though this outcome is very unlikely. This means that, like the 2-region model, the 3-region FE model exhibits a core-periphery pattern based on a "subcritical pitchfork" bifurcation. We have also concluded numerically that the dispersion of skilled workers among two regions is not sustainable in a model with three regions, where it corresponds to an outcome of partial dispersion.

All of these results are tantamount to those in the CP model with three regions in Castro *et al.* (2012), the difference being that, additionally, we were able to obtain explicit solutions for skilled wages and obtain relations between the relevant endogenous variables and the spatial distribution of skilled workers. Additionally, we proved that, when the manufacturing sector becomes irrelevant (resp. all local income is spent on agriculture) or it approaches perfect competition, migration decisions of skilled workers are the same in both the 2-region and the 3-region FE models. This, as was previously shown, occurs because there is convergence to zero of the critical values of the transport costs, where the stability of agglomeration and full dispersion changes. However, if this happens, concentration can never be a stable outcome, insofar as transport costs cannot fall below zero, while full dispersion, on the contrary, will always be a stable equilibrium. This come as no surprise, as a significant weight of the industrial sector and heterogeneity between the goods produced by it are among some of the factors essential to sustain a self-reinforced agglomeration process.

Although the FE model is able to give us closed form solutions, the assumptions it makes still do not allow enough simplification to fully assess analytically the dynamic properties when its analysis is applied to three regions. Although there is more relevant evidence in that outcomes such as partial dispersion can never be a stable outcome in a

Core Periphery model, nonlinearity in transport costs concerning its stability conditions still makes it impossible to analytically exclude the possibility that skilled workers might equally disperse across two regions when there are three regions available to migrate. Whereas the 2-region FE model is useful to address issues beyond the explanation capability of the original CP model by Krugman (1991b), doubts remain about whether it is suitable to tackle the more complex case of n regions, though such an analysis could be interesting, as the higher the number of regions subject to the study of Core-Periphery models, the better the insight we will get on New Economic Geography.

Appendix A

In order to study the dynamics of the Core-Periphery model with three regions it is necessary to find an expression for w_i, the reason being that skilled workers migrate to the region where they have the highest indirect utility, so naturally, a comparison between real wages is required.

Proof of proposition 2.1:

Proof. We have the following linear system of equations determining the nominal wages, after (2.13):

$$\begin{cases} w_1 = \frac{\mu}{\sigma} \sum_{j=1}^{3} \frac{\phi_{1j}\left(\frac{L}{3} + w_j H_j\right)}{R_j} \\ w_2 = \frac{\mu}{\sigma} \sum_{j=1}^{3} \frac{\phi_{2j}\left(\frac{L}{3} + w_j H_j\right)}{R_j} \\ w_3 = \frac{\mu}{\sigma} \sum_{j=1}^{3} \frac{\phi_{3j}\left(\frac{L}{3} + w_j H_j\right)}{R_j}, \end{cases}$$

which becomes, after some manipulation:

$$\begin{cases} w_1\left(1 - \frac{\mu}{\sigma}\frac{H_1}{R_1}\right) - w_2\left(\frac{\mu}{\sigma}\frac{\phi H_2}{R_2}\right) - w_3\left(\frac{\mu}{\sigma}\frac{\phi H_3}{R_3}\right) & = \frac{\mu}{\sigma}\frac{L}{3}\left(\frac{1}{R_1} + \frac{\phi}{R_2} + \frac{\phi}{R_3}\right) \\ w_1\left(-\frac{\mu}{\sigma}\frac{\phi H_1}{R_1}\right) + w_2\left(1 - \frac{\mu}{\sigma}\frac{H_2}{R_2}\right) - w_3\left(\frac{\mu}{\sigma}\frac{\phi H_3}{R_3}\right) & = \frac{\mu}{\sigma}\frac{L}{3}\left(\frac{\phi}{R_1} + \frac{1}{R_2} + \frac{\phi}{R_3}\right) \\ w_1\left(-\frac{\mu}{\sigma}\frac{\phi H_1}{R_1}\right) - w_2\left(\frac{\mu}{\sigma}\frac{\phi H_2}{R_2}\right) + w_3\left(1 - \frac{\mu}{\sigma}\frac{\phi H_3}{R_3}\right) & = \frac{\mu}{\sigma}\frac{L}{3}\left(\frac{\phi}{R_1} + \frac{\phi}{R_2} + \frac{1}{R_3}\right). \end{cases}$$

This may be written in matrix form as $AW = B$, where A stands for the coefficients matrix, W the vector of nominal wages w_i, while B is the column vector of independent terms in the right-hand side of the system of equations above. Applying Cramer's Rule, the solution to this system is of the following form:

$$w_i = \frac{Dw_i}{D},$$

where the denominator D stands for the determinant of matrix A and Dw_i is the determinant of the matrix obtained by replacing the i-th column of A by the column vector B. This method is useful since we only need to solve for a specific nominal wage, e.g w_1, and easily deduce the remaining solutions applying an argument of symmetry. Finding an expression for D first, we have:

$$
\begin{aligned}
D &= \left(1-\frac{\mu}{\sigma}\frac{H_1}{R_1}\right)\left(1-\frac{\mu}{\sigma}\frac{H_2}{R_2}\right)\left(1-\frac{\mu}{\sigma}\frac{H_3}{R_3}\right)+2\left(-\frac{\mu}{\sigma}\phi\frac{H_2}{R_2}\right)\left(-\frac{\mu}{\sigma}\phi\frac{H_3}{R_3}\right)\left(-\frac{\mu}{\sigma}\phi\frac{H_1}{R_1}\right)- \\
&-\left(1-\frac{\mu}{\sigma}\frac{H_1}{R_1}\right)\left(-\phi\frac{\mu}{\sigma}\frac{H_2}{R_2}\right)\left(-\frac{\mu}{\sigma}\phi\frac{H_3}{R_3}\right)-\left(-\frac{\mu}{\sigma}\phi\frac{H_1}{R_1}\right)\left(1-\frac{\mu}{\sigma}\frac{H_2}{R_2}\right)\left(-\frac{\mu}{\sigma}\phi\frac{H_3}{R_3}\right)- \\
&-\left(-\frac{\mu}{\sigma}\phi\frac{H_1}{R_1}\right)\left(-\frac{\mu}{\sigma}\phi\frac{H_2}{R_2}\right)\left(1-\frac{\mu}{\sigma}\frac{H_3}{R_3}\right) \\
&= 1-\frac{\mu}{\sigma}\left(\frac{H_1}{R_1}+\frac{H_2}{R_2}+\frac{H_3}{R_3}\right)+\frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left(\frac{H_1H_2}{R_1R_2}+\frac{H_1H_3}{R_1R_3}+\frac{H_2H_3}{R_2R_3}\right) \\
&-\frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\frac{H_1H_2H_3}{R_1R_2R_3} \\
\Leftrightarrow\quad D &= 1-\frac{\mu}{\sigma}\sum_{j=1}^{3}\frac{H_j}{R_j}+\frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left(\frac{H_1H_2}{R_1R_2}+\frac{H_1H_3}{R_1R_3}+\frac{H_2H_3}{R_2R_3}\right)- \\
&-\frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\prod_{j=1}^{3}\frac{H_j}{R_j}, \qquad (5.1)
\end{aligned}
$$

which is obviously invariant under any distribution of skilled workers across the regions, since it is a common denominator for every solution of the nominal wage w_i.

The numerator of w_1, Dw_1, can be calculated as follows:

$$
\begin{aligned}
Dw_1 &= \frac{\mu}{\sigma}\frac{L}{3}a\left(1-\frac{\mu}{\sigma}\frac{H_2}{R_2}\right)\left(1-\frac{\mu}{\sigma}\frac{H_3}{R_3}\right)+\frac{\mu}{\sigma}\frac{L}{3}c\left(\frac{\mu}{\sigma}\phi\frac{H_3}{R_3}\right)\left(\frac{\mu}{\sigma}\phi\frac{H_2}{R_2}\right)+\frac{\mu}{\sigma}\frac{L}{3}b\left(\frac{\mu}{\sigma}\phi\frac{H_2}{R_2}\right)\left(\frac{\mu}{\sigma}\phi\frac{H_3}{R_3}\right) \\
&-\frac{\mu}{\sigma}\frac{L}{3}c\left(1-\frac{\mu}{\sigma}\frac{H_2}{R_2}\right)\left(-\frac{\mu}{\sigma}\phi\frac{H_3}{R_3}\right)-\frac{\mu}{\sigma}\frac{L}{3}a\left(\frac{\mu}{\sigma}\phi\frac{H_2}{R_2}\right)\left(\frac{\mu}{\sigma}\phi\frac{H_3}{R_3}\right)-\frac{\mu}{\sigma}\frac{L}{3}b\left(-\frac{\mu}{\sigma}\phi\frac{H_2}{R_2}\right)\left(1-\frac{\mu}{\sigma}\frac{H_3}{R_3}\right) \\
&= \frac{\mu}{\sigma}\frac{L}{3}a\left[1-\frac{\mu}{\sigma}\left(\frac{H_2}{R_2}+\frac{H_3}{R_3}\right)\right]+\frac{\mu^2}{\sigma^2}\frac{L}{3}\phi\left(b\frac{H_2}{R_2}+c\frac{H_3}{R_3}\right)+ \\
&+\left(a+c\phi^2+b\phi^2-c\phi-a\phi^2-b\phi\right)\frac{\mu^3}{\sigma^3}\frac{L}{3}\frac{H_2H_3}{R_2R_3}, \\
\Leftrightarrow\quad Dw_1 &= \frac{\mu}{\sigma}\frac{L}{3}\left\{a+\frac{\mu}{\sigma}\left[\frac{H_2}{R_2}\left(\phi b-a\right)+\frac{H_3}{R_3}\left(\phi c-a\right)\right]+\left(a+c\phi^2+b\phi^2-c\phi-a\phi^2-b\phi\right)\frac{\mu^3}{\sigma^3}\frac{H_2H_3}{R_2R_3}\right\}
\end{aligned}
$$

where $a=\left(\frac{1}{R_1}+\frac{\phi}{R_2}+\frac{\phi}{R_3}\right)$,$b=\left(\frac{\phi}{R_1}+\frac{1}{R_2}+\frac{\phi}{R_3}\right)$ and $c=\left(\frac{\phi}{R_1}+\frac{\phi}{R_2}+\frac{1}{R_3}\right)$. Simplifying to eliminate a, b and c, we end up with:

$$
\begin{aligned}
Dw_1 &= \frac{\mu}{\sigma}\frac{L}{3}\left\{\left(\frac{1}{R_1}+\frac{\phi}{R_2}+\frac{\phi}{R_3}\right)+\frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{H_2+H_3}{R_2R_3}+\frac{\phi^2-1}{R_1}\left(\frac{H_2}{R_2}+\frac{H_3}{R_3}\right)\right]+\right.\\
&\left.+\frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{H_2H_3}{R_1R_2R_3}\right\}\\
\Leftrightarrow\quad Dw_1 &= \frac{\mu}{\sigma}\frac{L}{3}\left\{\left(\sum_{j=1}^{3}\frac{\phi_{1j}}{R_j}\right)+\frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{H_2+H_3}{R_2R_3}+\frac{\phi^2-1}{R_1}\left(\frac{H_2}{R_2}+\frac{H_3}{R_3}\right)\right]+\right.\\
&\left.+\frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{H_2H_3}{R_1R_2R_3}\right\} \qquad (5.2)
\end{aligned}
$$

The expression for the nominal wage in region 1 is obtained dividing equation (5.2) by equation (5.1):

$$
w_1=\frac{\frac{\mu}{\sigma}\frac{L}{3}\left\{\sum_{j=1}^{3}\frac{\phi_{1j}}{R_j}+\frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{H_2+H_3}{R_2R_3}+\frac{\phi^2-1}{R_1}\left(\frac{H_2}{R_2}+\frac{H_3}{R_3}\right)\right]+\frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{H_2H_3}{R_1R_2R_3}\right\}}{1-\frac{\mu}{\sigma}\sum_{j=1}^{3}\frac{H_j}{R_j}+\frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left(\frac{H_1H_2}{R_1R_2}+\frac{H_1H_3}{R_1R_3}+\frac{H_2H_3}{R_2R_3}\right)-\frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\prod_{j=1}^{3}\frac{H_j}{R_j}}.
$$

One can see that, under a given distribution of H, $w_1(H_1,H_2,H_3)=w_1(H_1,H_3,H_2)$, which is a consequence of the existing symmetry in region 2 and region 3, since there is nothing to distinguish between the two regions. An analogous argument can be made concerning region 1 and any of the other two regions. This means that the nominal wage in region i is invariant in the distribution of skilled workers in the other two regions. Symmetry among the regions also asserts that $w_1\left(H_1,H_2,H_3\right)=w_2\left(H_2,H_1,H_3\right)=w_3\left(H_3,H_1,H_2\right)$. Thus, we can formulate a general expression for the numerator of the nominal wage, Dw_i:

$$
\begin{aligned}
Dw_i &= \frac{\mu}{\sigma}\frac{L}{3}\left\{\sum_{j=1}^{3}\frac{\phi_{ij}}{R_j}+\frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{\sum_{k\neq i}H_k}{\prod_{k\neq i}R_k}+\frac{\phi^2-1}{R_i}\sum_{k\neq i}\frac{H_k}{R_k}\right]+\right.\\
&\left.+\frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{1}{R_i}\prod_{k\neq i}\frac{H_k}{R_k}\right\},\quad i=\{1,2,3\}.
\end{aligned}
$$

It follows that the nominal wage w_i is the quotient between the latter equation and the determinant D in (5.1):

$$w_i = \frac{\frac{\mu}{\sigma}\frac{L}{3}\left\{\sum_{j=1}^{3}\frac{\phi_{1j}}{R_j} + \frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{\sum_{k\neq i} H_k}{\prod_{k\neq i} R_k} + \frac{\phi^2-1}{R_i}\sum_{k\neq i}\frac{H_k}{R_k}\right] + \frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{1}{R_i}\prod_{k\neq i}\frac{H_k}{R_k}\right\}}{1-\frac{\mu}{\sigma}\sum_{j=1}^{3}\frac{H_j}{R_j} + \frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left(\frac{H_1H_2}{R_1R_2}+\frac{H_1H_3}{R_1R_3}+\frac{H_2H_3}{R_2R_3}\right) - \frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\prod_{j=1}^{3}\frac{H_j}{R_j}}.$$

□

Appendix B

Proof of proposition 3.9:

Proof. We have at most three different types of configurations $(h_1, h_2, 1-h_1-h_2)$ when $h_i = \dfrac{1-h_j}{2}$ and $0 < h_i < 1$.

First, if $0 < h_1 = \dfrac{1-h_2}{2} < 1$, we have:

$$\left(\frac{1-h_2}{2}, h_2, \frac{1-h_2}{2}\right) = (a, b, a).$$

Second, we have $0 < h_2 = \dfrac{1-h_2}{2} < 1$:

$$\left(h_1, \frac{1-h_2}{2}, \frac{1-h_2}{2}\right) = (b, a, a).$$

Finally, if $h_2 = h_1$, we end up with:

$$\left(\frac{1-h_2}{2}, \frac{1-h_2}{2}, \frac{1-h_2}{2}\right) = (a, a, a).$$

Consider the first case, where $h_1 = a = \dfrac{1-h_2}{2}$. It follows that

$$\bar{\omega}(h_1, h_2, 1-h_1-h_2) = \bar{\omega}(a, b, a).$$

If $\dfrac{\partial \bar{\omega}}{\partial h_1} > 0$, then:

$$\exists \varepsilon > 0 \, : \, \bar{\omega}(a+\varepsilon, b, a-\varepsilon) > \bar{\omega}(a, b, a) \, ^1.$$

However, symmetry among regions means the average real wage is invariant after coordinate permutation, implying:

$$\bar{\omega}(a+\varepsilon, b, a-\varepsilon) = \bar{\omega}(a-\varepsilon, b, a+\varepsilon),$$

and hence:

$$\bar{\omega}(a-\varepsilon, b, a+\varepsilon) > \bar{\omega}(a, b, a),$$

contradicting $\frac{\partial \bar{\omega}}{\partial h_1} > 0$. By analogy, it is also false that $\frac{\partial \bar{\omega}}{\partial h_1} < 0$. Therefore, $\frac{\partial \bar{\omega}}{\partial h_1} = 0$. An analogous reasoning yields $\frac{\partial \bar{\omega}}{\partial h_2} = 0$. $\square$

Appendix C

This Appendix includes the calculations of the partial derivatives and their value at the relevant configurations required to find the conditions for stability of total dispersion and partial dispersion, respectively. It is composed of some very tedious algebra, so uninterested readers can skip right to part C.1 or C.2 of the Appendix unconcernedly.

To find all necessary derivatives, it is best to do this in a stepwise fashion, applying

[1] Recall that h_3 is a function of h_1 and h_2. Since we are working in two coordinates and h_3 is implicitly defined, the concept of partial derivative means that h_2 should remain constant when we vary h_1. Every variation in h_1 must therefore be reflected in h_3 and we write it explicitly so that the notion and consequences of symmetry are made clearer.

quotient rules successively. Let us begin by differentiating ω_i with respect to h_i:

$$\frac{\partial \omega_i}{\partial h_i} = \frac{\dfrac{\partial w_i}{\partial h_i} P_1^{\mu} - \dfrac{\partial P_i^{\mu}}{\partial h_i} w_1}{P_i^{2\mu}},$$

and in its turn:

$$\frac{\partial w_i}{\partial h_i} = \frac{\dfrac{\partial D w_i}{\partial h_i} D - \dfrac{\partial D}{\partial h_i} D w_i}{D^2}.$$

This is easier if we do this from the perspective of a specific region, say region 1 and differentiate with respect to h_1 and h_2. We can then generalize for any region i, considering symmetry between regions. Starting with $\frac{\partial w_1}{\partial h_1}$, picking first Dw_1 from (2.16):

$$\begin{aligned}
\frac{\partial D w_1}{\partial h_1} &= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}\left\{\frac{\phi-1}{{r_1}^2} + \frac{(1-\phi)\phi}{{r_3}^2} + \frac{\mu}{\sigma}(\phi-1)\left[-\frac{(\phi-1)\phi(1-h_1)}{r_2 r_3^2} - \frac{\phi}{r_2 r_3} + (\phi+1)\left(-\frac{(-1+\phi)h_3}{r_1 r_3^2} - \frac{1}{r_1 r_3}\right) - \right.\right.\\
&\left.\left. -(1-\phi)(\phi+1)\left(\frac{h_2}{r_2 r_1^2} + \frac{h_3}{r_3 r_1^2}\right)\right] + \frac{\mu^2}{\sigma^2}\left(1-3\phi^2+2\phi^3\right)\frac{h_2}{r_2}\left(\frac{(\phi-1)h_3}{r_1 r_3^2} - \frac{1}{r_1 r_3} + \frac{(\phi-1)h_3}{r_1^2 r_3}\right)\right\} \Leftrightarrow \\
\frac{\partial D w_1}{\partial h_1} &= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}\left\{\frac{\phi-1}{{r_1}^2} + \frac{(1-\phi)\phi}{{r_3}^2} + \frac{\mu}{\sigma}\left[-\frac{(\phi-1)\phi(1-h_1)}{r_2 r_3^2} - \frac{\phi}{r_2 r_3} - \frac{\phi(\phi+1)}{r_1 r_3^2} + \left(\phi^2-1\right)\frac{1}{r_1^2}\left(\frac{h_2}{r_2} + \frac{h_3}{r_3}\right)\right] + \right.\\
&\left. +\frac{\mu^2}{\sigma^2}\left(1-3\phi^2+2\phi^3\right)\frac{h_2}{r_2}\left(\frac{(\phi-1)h_3}{r_1^2 r_3} - \frac{\phi}{r_1 r_3^2}\right)\right\}
\end{aligned}$$

Further simplification renders:

$$\begin{aligned}
\frac{\partial D w_1}{\partial h_1} &= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}(\phi-1)\left\{\frac{1}{{r_1}^2} - \frac{\phi}{{r_3}^2} + \frac{\mu}{\sigma}\left[-\phi\left(\frac{\phi-h_2+\phi h_2}{r_2 r_3^2} + \frac{\phi+1}{r_1 r_3^2}\right) + \left(\phi^2-1\right)\frac{1}{r_1^2}\left(\frac{h_2}{r_2} + \frac{h_3}{r_3}\right)\right] + \right.\\
&\left. +\frac{\mu^2}{\sigma^2}\left(2\phi^2-\phi-1\right)\frac{h_2}{r_2}\left(\frac{(\phi-1)h_3}{r_1^2 r_3} - \frac{\phi}{r_1 r_3^2}\right)\right\}.
\end{aligned}$$

Symmetry between regions, whereby $w_1(h_1, h_2) = w_2(h_2, h_1)$ implies $\frac{\partial D w_2}{\partial h_2}(h_2, h_1) = \frac{\partial D w_1}{\partial h_1}(h_1, h_2)$. Thus, the following is true:

$$\begin{aligned}
\frac{\partial D w_i}{\partial h_i} &= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}(\phi-1)\left\{\frac{1}{{r_i}^2} - \frac{\phi}{{r_3}^2} + \frac{\mu}{\sigma}\left[-\phi\left(\frac{\phi-h_j+\phi h_j}{r_j r_k^2} + \frac{\phi+1}{r_i r_k^2}\right) + \left(\phi^2-1\right)\frac{1}{r_i^2}\left(\frac{h_j}{r_j} + \frac{h_k}{r_k}\right)\right] + \right.\\
&\left. +\frac{\mu^2}{\sigma^2}\left(2\phi^2-\phi-1\right)\frac{h_j}{r_j}\left(\frac{(\phi-1)h_k}{r_i^2 r_k} - \frac{\phi}{r_i r_k^2}\right)\right\}, \quad i,j,k = (1,2,3),
\end{aligned}$$

where

$$r_k = 1 + (\phi-1)(h_i + h_j) \text{ and } h_k = 1 - h_i - h_j.$$

Moving on to $\dfrac{\partial D}{\partial h_1}$, through (2.17) we have:

$$\begin{aligned}
\frac{\partial D}{\partial h_1} &= -\frac{\mu}{\sigma}\phi\left(\frac{1}{r_1^2}-\frac{1}{r_3^2}\right)+\frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left[-\frac{(1-\phi)h_1h_2}{r_1^2r_2}+\frac{h_2}{r_1r_2}-\frac{(-1+\phi)h_1h_3}{r_1r_3^2}\right.\\
&\left.-\frac{(-1+\phi)h_3h_2}{r_2r_3^2}-\frac{h_1}{r_1r_2}-\frac{(1-\phi)h_1h_3}{r_1r_3^2}+\frac{h_3}{r_1r_3}-\frac{h_2}{r_2r_3}\right]-\frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\times\\
&\times\frac{h_2}{r_2}\left[-\frac{(-1+\phi)h_1h_3h_2}{r_1r_2r_3^2}-\frac{h_1h_2}{r_1r_2r_3}-\frac{(1-\phi)h_1h_3h_2}{r_1^2r_2r_3}+\frac{h_3h_2}{r_1r_2r_3}\right],
\end{aligned}$$

which becomes, after more manipulation:

$$\begin{aligned}
\frac{\partial D}{\partial h_1} &= -\frac{\mu}{\sigma}\phi\left(\frac{1}{r_1^2}-\frac{1}{r_3^2}\right)+\frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left[\phi\frac{(-1+2h_1+h_2)\left(-\phi+2(-1+\phi)h_2\left(1+(-1+\phi)h_2\right)\right)}{r_1^2r_2r_3^2}\right]\\
&+\frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\frac{h_2}{r_2}\left[\phi\frac{h_2\left(-1+2h_1+h_2\right)\left(1+(-1+\phi)h_2\right)}{r_1^2r_2r_3^2}\right]\Leftrightarrow\\
\frac{\partial D}{\partial h_1} &= \frac{\mu}{\sigma}\phi\left\{-\frac{1}{r_1^2}+\frac{1}{r_3^2}+\frac{\mu}{\sigma}(\phi-1)\frac{2h_1+h_2-1}{r_1^2r_2r_3^2}\left[-(\phi+1)\left(-\phi+2(-1+\phi)h_2\left(1+(-1+\phi)h_2\right)\right)+\right.\right.\\
&\left.\left.+\frac{\mu^2}{\sigma^2}\left(2\phi^2-\phi-1\right)\frac{h_2^2}{r_2}\left(1+(-1+\phi)h_2\right)\right]\right\}.
\end{aligned}$$

Furthermore, because of symmetry between regions, the derivatives of D with respect to h_i can be expressed in the following fashion:

$$\begin{aligned}
\frac{\partial D}{\partial h_i} &= \frac{\mu}{\sigma}\phi\left\{-\frac{1}{r_i^2}+\frac{1}{r_k^2}+\frac{\mu}{\sigma}(\phi-1)\phi\frac{2h_i+h_j-1}{r_i^2r_jr_k^2}\left[(1-\phi)\left(-\phi+2(-1+\phi)h_i\left(1+(-1+\phi)h_i\right)\right)+\right.\right.\\
&\left.\left.+\frac{\mu^2}{\sigma^2}\left(2\phi^2-\phi-1\right)\frac{h_j^2}{r_j}\left(1+(-1+\phi)h_j\right)\right]\right\},\quad i,j,k=(1,2,3)\,. \qquad (5.3)
\end{aligned}$$

As for $\dfrac{\partial Dw_1}{\partial h_2}$, it goes:

$$\begin{aligned}
\frac{\partial Dw_1}{\partial h_2} &= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left(\frac{\phi(\phi-1)}{r_2^2}+\frac{\phi(1-\phi)}{r_3^2}\right)+\frac{\mu^2}{\sigma^2}\frac{L}{3}\frac{1}{H}(\phi-1)\left[-\frac{(-1+\phi)\phi(1-h_1)}{r_2r_3^2}-\right.\\
&\left.-\frac{(1-\phi)\phi(1-h_1)}{r_2^2r_3}+(1+\phi)\left(-\frac{(1-\phi)h_2}{r_1r_2^2}+\frac{1}{r_1r_2}-\frac{(\phi-1)h_3}{r_1r_3^2}-\frac{1}{r_1r_3}\right)\right]+\\
&+\frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{1}{r_1}\left[-\frac{(-1+\phi)h_2h_3}{r_2r_3^2}-\frac{(1-\phi)h_2h_3}{r_2^2r_3}+\frac{h_3}{r_2r_3}-\frac{h_2}{r_2r_3}\right]
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial Dw_1}{\partial h_2} &= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left(\frac{\phi\left(\phi-1\right)}{r_2^2}+\frac{\phi\left(1-\phi\right)}{r_3^2}\right)+\frac{\mu^2}{\sigma^2}\frac{L}{3}\frac{1}{H}\left(\phi-1\right)\times\\
&\times\frac{(\phi-1)\phi\left(1+\phi+2\phi^2-\left(2-3\phi+\phi^2\right)h_1+(-1+\phi)^2h_1^2\right)\left(-1+h_1+2h_2\right)}{r_1r_2^2r_3^2}+\\
&+\frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{1}{r_1}\left[-\frac{\phi\left(1+(-1+\phi)h_1\right)\left(-1+h_1+2h_2\right)}{r_2^2r_3^2}\right]\Leftrightarrow\\
\Leftrightarrow\frac{\partial Dw_1}{\partial h_2} &= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\phi\left(\phi-1\right)\left\{\frac{1}{r_2^2}-\frac{1}{r_3^2}+\frac{\mu}{\sigma}\frac{h_1+2h_2-1}{r_1r_2^2r_3^2}\left(\phi-1\right)\times\right.\\
&\times\left[\left(1+\phi+2\phi^2-\left(2-3\phi+\phi^2\right)h_1+(-1+\phi)^2h_1^2\right)-\right.\\
&\left.\left.-\frac{\mu^2}{\sigma^2}\left(2\phi+1\right)\left(1+(-1+\phi)h_1\right)\right]\right\}.
\end{aligned}
$$

Once again, symmetry between regions imposes $\dfrac{\partial Dw_1}{\partial h_2}(h_1,h_2) = \dfrac{\partial Dw_2}{\partial h_1}(h_2,h_1)$, so generalization for any region i implies:

$$
\begin{aligned}
\frac{\partial Dw_i}{\partial h_j} &= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\phi\left(\phi-1\right)\left\{\frac{1}{r_j^2}-\frac{1}{r_k^2}+\frac{\mu}{\sigma}\frac{h_i+2h_j-1}{r_ir_j^2r_k^2}\left(\phi-1\right)\left[\left(1+\phi+2\phi^2-\right.\right.\right.\\
&\left.\left.\left.-\left(2-3\phi+\phi^2\right)h_i+(-1+\phi)^2h_i^2\right)-\frac{\mu^2}{\sigma^2}\left(2\phi+1\right)\left(1+(-1+\phi)h_i\right)\right]\right\}.
\end{aligned}
$$

Having found the expressions for all derivatives that compose $\frac{\partial w_i}{\partial h_i}$, we now have only to find a general expression for the derivatives of P_i^{μ}, so we can use it to obtain the values of $\frac{\partial \omega_i}{\partial h_i}$ in the different configurations. This can be done effortlessly for any region i, differentiating P_i^{μ} with respect to h_i considering P_i in equation (2.18):

$$
\frac{\partial P_i^{\mu}}{\partial h_i} = \mu\frac{1-\phi}{1-\sigma}\left[\beta\frac{\sigma}{\sigma-1}\left(\frac{H}{\alpha}\right)^{\frac{1}{1-\sigma}}\right]^{\mu} r_i^{\frac{1}{1-\sigma}-1}.
$$

C.1 Total Dispersion

Proof of Proposition 3.12:

Proof. The configuration $(H_1,H_2,H_3) = \left(\frac{H}{3},\frac{H}{3},\frac{H}{3}\right)$ is equivalent to $(h_1,h_2) = \left(\frac{1}{3},\frac{1}{3}\right)$. At this point, we denote:

$$
r_1, r_2, r_3 = \tfrac{2\phi+1}{3} = r.
$$

for simplification purposes.

Total dispersion is stable as long as:

$$\begin{aligned}
\frac{\partial \omega_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) &< 0 \Leftrightarrow \\
\frac{\partial w_1}{\partial h_1}P_1^{\mu} - \frac{\partial P_1^{\mu}}{\partial h_1}w_1 &< 0,
\end{aligned}$$

since the price index has to be positive $\left(P_1^{2\mu} > 0\right)$.

Substitution of $\left(\tfrac{1}{3},\tfrac{1}{3}\right)$ in (5.3) yields $\frac{\partial D}{\partial h_i}\left(\tfrac{1}{3},\tfrac{1}{3}\right) = 0$, since:

$$2h_i + h_j - 1 = \frac{2}{3} + \frac{1}{3} - 1 = 0 \text{ and } -\frac{1}{r_1^2} + \frac{1}{r_2^2} = -\frac{1}{r^2} + \frac{1}{r^2} = 0.$$

This enables us to write $\dfrac{\partial w_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right)$ as follows:

$$\frac{\partial w_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) = \frac{\partial D w_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) D^{-1}\left(\tfrac{1}{3},\tfrac{1}{3}\right).$$

However, remark that:

$$\begin{gathered}
\frac{\partial w_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) P_1^{\mu}\left(\tfrac{1}{3},\tfrac{1}{3}\right) - \frac{\partial P_1^{\mu}}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) w_1\left(\tfrac{1}{3},\tfrac{1}{3}\right) < 0 \Leftrightarrow \\
\Leftrightarrow \frac{\partial P_1^{\mu}}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) P_1^{-\mu}\left(\tfrac{1}{3},\tfrac{1}{3}\right) > \frac{\partial w_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) w_1^{-1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) \Leftrightarrow \\
\Leftrightarrow \frac{\partial P_1^{\mu}}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) P_1^{-\mu}\left(\tfrac{1}{3},\tfrac{1}{3}\right) > \frac{\partial D w_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) D^{-1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) D\left(\tfrac{1}{3},\tfrac{1}{3}\right) D w_1^{-1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) \Leftrightarrow \\
\Leftrightarrow \frac{\partial D w_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) D w_1^{-1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) < \frac{\partial P_1^{\mu}}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) P_1^{-\mu}\left(\tfrac{1}{3},\tfrac{1}{3}\right). \qquad (5.4)
\end{gathered}$$

where:

$$
\begin{aligned}
\frac{\partial Dw_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) &= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}(\phi-1)\left\{\frac{1-\phi}{r^2}+\frac{\mu}{\sigma}\frac{1}{r^3}\left[-\phi\left(\phi-\frac{1}{3}+\phi\frac{1}{3}+\phi+1\right)+\frac{2}{3}\left(\phi^2-1\right)\right]+\right.\\
&\quad\left.+\frac{\mu^2}{\sigma^2}\left(2\phi^2-\phi-1\right)\frac{1}{3}\frac{1}{r^4}\left(\frac{1}{3}(\phi-1)-\phi\right)\right\}\Leftrightarrow\\
\Leftrightarrow\frac{\partial Dw_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) &= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}(\phi-1)\left\{\frac{1-\phi}{r^2}-\frac{1}{3}\frac{\mu}{\sigma}\frac{1}{r^3}\left[\left(2+2\phi+5\phi^2\right)\right]+\right.\\
&\quad\left.-\frac{\mu^2}{\sigma^2}(\phi-1)\frac{(2\phi+1)}{3}\frac{1}{r^4}\frac{(2\phi+1)}{3}\right\}\\
&= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}(\phi-1)\left\{\frac{1-\phi}{r^2}-\frac{1}{3}\frac{\mu}{\sigma}\frac{1}{r^3}\left(2+2\phi+5\phi^2\right)+\frac{\mu^2}{\sigma^2}(1-\phi)\frac{1}{r^2}\right\}\\
\Leftrightarrow\frac{\partial Dw_1}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right) &= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}(\phi-1)\frac{1}{r^2}\left\{(1-\phi)\left(1+\frac{\mu^2}{\sigma^2}\right)-\frac{1}{3}\frac{\mu}{\sigma}\frac{1}{r}\left(2+2\phi+5\phi^2\right)\right\}.
\end{aligned}
$$

As for $Dw_1\left(\frac{1}{3},\frac{1}{3}\right)$, we have:

$$
\begin{aligned}
Dw_1\left(\tfrac{1}{3},\tfrac{1}{3}\right) &= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left\{\frac{2\phi+1}{r}+\frac{\mu}{\sigma}\frac{2}{3}(\phi-1)(2\phi+1)\frac{1}{r^2}+\frac{\mu^2}{\sigma^2}(2\phi+1)(\phi-1)^2\frac{1}{9}\frac{1}{r^3}\right\}\\
&= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}(2\phi+1)\left\{\frac{1}{r}+\frac{\mu}{\sigma}\frac{2}{3}(\phi-1)\frac{1}{r^2}+\frac{\mu^2}{\sigma^2}(\phi-1)^2\frac{1}{9r^3}\right\}\Leftrightarrow\\
&= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left\{3+\frac{\mu}{\sigma}2(\phi-1)\frac{1}{r}+\frac{\mu^2}{\sigma^2}(\phi-1)^2\frac{1}{3r^2}\right\}\Leftrightarrow\\
\Leftrightarrow Dw_1\left(\tfrac{1}{3},\tfrac{1}{3}\right) &= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left\{3+\frac{\phi-1}{r}\left[2\frac{\mu}{\sigma}+\frac{\mu^2}{\sigma^2}\frac{(\phi-1)}{3r}\right]\right\}.
\end{aligned}
$$

Moreover, notice that:

$$
\frac{\partial P_1^{\mu}}{\partial h_1}\left(\tfrac{1}{3},\tfrac{1}{3}\right)P_1^{-\mu}\left(\tfrac{1}{3},\tfrac{1}{3}\right)=\frac{\mu}{r}\frac{1-\phi}{1-\sigma}.
$$

We can now finally compute the expression for stability of total dispersion implicit in (5.4):

$$
\begin{aligned}
&\frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}(\phi-1)\frac{1}{r^2}\left\{(1-\phi)\left(1+\frac{\mu^2}{\sigma^2}\right)-\frac{1}{3}\frac{\mu}{\sigma}\frac{1}{r}\left(2+2\phi+5\phi^2\right)\right\}<\\
&\qquad<\frac{\mu}{r}\frac{1-\phi}{1-\sigma}\frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left\{3+\frac{\phi-1}{r}\left[2\frac{\mu}{\sigma}+\frac{\mu^2}{\sigma^2}\frac{(\phi-1)}{3r}\right]\right\}\Leftrightarrow
\end{aligned}
$$

$$
\begin{aligned}
&\Leftrightarrow\frac{\phi-1}{r}\left\{(1-\phi)\left(1+\frac{\mu^2}{\sigma^2}\right)-\frac{1}{3}\frac{\mu}{\sigma}\frac{1}{r}\left(2+2\phi+5\phi^2\right)\right\}<\\
&\qquad<-\mu\frac{\phi-1}{\sigma-1}\left\{3+\frac{(\phi-1)}{r}\left[2\frac{\mu}{\sigma}+\frac{\mu^2}{\sigma^2}\frac{(\phi-1)}{3r}\right]\right\},
\end{aligned}
$$

thus yielding:

$$(\phi-1)\left\{\frac{1}{2\phi+1}\left[(1-\phi)\left(1+\frac{\mu^2}{\sigma^2}\right)-\frac{\mu}{\sigma}\frac{2+2\phi+5\phi^2}{2\phi+1}\right]+\frac{\mu}{1-\sigma}\left[1+\frac{\phi-1}{2\phi+1}\left(2\frac{\mu}{\sigma}+\frac{\mu^2}{\sigma^2}\frac{\phi-1}{2\phi+1}\right)\right]\right\} \quad < \quad 0.$$

Rewriting the inequality after placing the expression on the left-hand side under the common denominator $(-1+\sigma)(\sigma+2\sigma\phi)^2$ entails:

$$\begin{aligned}
\frac{(1-\phi)(\sigma+\mu(-1+\phi)+2\sigma\phi)\left(\mu^2(-1+\phi)+(-1+\sigma)\sigma(-1+\phi)+\mu(-1+2\sigma)(1+2\phi)\right)}{(-1+\sigma)(\sigma+2\sigma\phi)^2} &< 0 \Leftrightarrow \\
(1-\phi)\left(\mu^2(-1+\phi)+(-1+\sigma)\sigma(-1+\phi)+\mu(-1+2\sigma)(1+2\phi)\right) &< 0 \Leftrightarrow \\
\mu^2(\phi-1)+(\sigma-1)\sigma(\phi-1)+\mu(-1+2\sigma)(1+2\phi) &< 0.
\end{aligned}$$

□

C.2 Partial Dispersion

Proof of Proposition 3.17:

Proof. As our model is specified, the configuration $(H_1, H_2, H_3) = \left(0, \frac{H}{2}, \frac{H}{2}\right)$ is the same as $(h_1, h_2) = \left(0, \frac{1}{2}\right)$. This means region 1 is the chosen to be left out of skilled workers, so the endowment H is equally distributed across regions 2 and 3. First of all, note the implications this has on r_i. We have:

$$r_1 = \phi \text{ and } r_2, r_3 = \frac{1+\phi}{2}.$$

Stability of this equilibrium requires ξ and β in to be negative. It has been shown in Section 3.3 that the stability conditions pertain to:

$$\frac{\partial \omega_2}{\partial h_2}\left(0, \tfrac{1}{2}\right) \quad < \quad 0,$$

corresponding to $\beta < 0$, and:

$$\omega_1\left(0, \tfrac{1}{2}\right) < \omega_2\left(0, \tfrac{1}{2}\right),$$

which refers to $\xi < 0$.

The first condition states that:

$$\frac{\partial w_2}{\partial h_2}\left(0,\tfrac{1}{2}\right)P_2^{\mu}\left(0,\tfrac{1}{2}\right)-\frac{\partial P_2^{\mu}}{\partial h_2}\left(0,\tfrac{1}{2}\right)w_2\left(0,\tfrac{1}{2}\right)<0,$$

since price indexes have to be positive. Recalling $\frac{\partial w_i}{\partial h_i}$, it follows:

$$\frac{\partial w_2}{\partial h_2}\left(0,\tfrac{1}{2}\right)=\frac{\dfrac{\partial Dw_2}{\partial h_2}\left(0,\tfrac{1}{2}\right)D\left(0,\tfrac{1}{2}\right)-\dfrac{\partial D}{\partial h_2}\left(0,\tfrac{1}{2}\right)Dw_2\left(0,\tfrac{1}{2}\right)}{D^2\left(0,\tfrac{1}{2}\right)}.$$

After substitution of $\left(0,\tfrac{1}{2}\right)$ in $\frac{\partial D}{\partial h_2}\left(0,\tfrac{1}{2}\right)$, rendering it equal to zero, we have:

$$\frac{\partial w_2}{\partial h_2}\left(0,\tfrac{1}{2}\right)=\frac{\partial Dw_2}{\partial h_2}\left(0,\tfrac{1}{2}\right)D^{-1}\left(0,\tfrac{1}{2}\right).$$

Hence, we are able to rewrite the condition for stability of partial dispersion in the same fashion as we did for total dispersion:

$$\frac{\partial Dw_2}{\partial h_2}\left(0,\tfrac{1}{2}\right)Dw_2^{-1}\left(0,\tfrac{1}{2}\right)<\frac{\partial P_2^{\mu}}{\partial h_2}\left(0,\tfrac{1}{2}\right)P_2^{-\mu}\left(0,\tfrac{1}{2}\right).$$

First, we have:

$$\begin{aligned}
\frac{\partial Dw_2}{\partial h_2}\left(0,\tfrac{1}{2}\right) &= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}(\phi-1)\left\{\frac{1-\phi}{r^2}+\frac{\mu}{\sigma}\left[-\frac{\phi}{r^2}-2\frac{\phi}{r^2}+\frac{\phi-1}{r^2}\right]\right\}= \\
&= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}(\phi-1)\left\{4\frac{1-\phi}{(1+\phi)^2}-\frac{\mu}{\sigma}\left[4\frac{2\phi+2}{(1+\phi)^2}\right]\right\}\Leftrightarrow \\
\Leftrightarrow \frac{\partial Dw_2}{\partial h_2}\left(0,\tfrac{1}{2}\right) &= \frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}\frac{1}{r^2}(\phi-1)\left[(1-\phi)-\frac{\mu}{\sigma}(2\phi+2)\right].
\end{aligned}$$

As for Dw_2, it goes:

$$\begin{aligned}
Dw_2\left(0,\tfrac{1}{2}\right) &= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left\{1+\frac{1+\phi}{r}+\frac{\mu}{\sigma}\left[\frac{1}{2}(\phi-1)\frac{1}{r}+\frac{1}{2}\frac{\phi^2-1}{r^2}\right]\right\} \\
&= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left\{3+\frac{1}{2}\frac{\mu}{\sigma}\left[(\phi-1)\frac{1}{r}+\frac{(\phi-1)(\phi+1)}{r^2}\right]\right\} \\
&= \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left\{3+\frac{\mu}{\sigma}3\frac{\phi-1}{\phi+1}\right\} \Leftrightarrow \\
\Leftrightarrow Dw_2\left(0,\tfrac{1}{2}\right) &= \frac{\mu}{\sigma}\frac{L}{H}\left[1+\frac{\mu}{\sigma}\frac{\phi-1}{\phi+1}\right].
\end{aligned}$$

Lastly, it is straightforward to see that:

$$\frac{\partial P_2^{\mu}}{\partial h_2}\left(0,\tfrac{1}{2}\right)P_2^{-\mu}\left(0,\tfrac{1}{2}\right)=\frac{\mu}{r}\frac{1-\phi}{1-\sigma}.$$

Thus we can finally determine the condition $\beta<0$:

$$\begin{gathered}
\beta<0 \Leftrightarrow \\
\frac{1}{H}\frac{\mu}{\sigma}\frac{L}{3}\frac{1}{r^2}(\phi-1)\left[(1-\phi)-\frac{\mu}{\sigma}(2\phi+2)\right]<\frac{\mu}{r}\frac{1-\phi}{1-\sigma}\frac{\mu}{\sigma}\frac{L}{H}\left[1+\frac{\mu}{\sigma}\frac{\phi-1}{\phi+1}\right] \Leftrightarrow \\
4\frac{\phi-1}{(\phi+1)^2}\left[(1-\phi)-\frac{\mu}{\sigma}(2\phi+2)\right]<\frac{\mu}{1+\phi}\frac{1-\phi}{1-\sigma}\left[1+\frac{\mu}{\sigma}\frac{\phi-1}{\phi+1}\right].
\end{gathered}$$

Placing everything under a common denominator yields, after some manipulation:

$$-\frac{2L\mu(-1+\phi)\left(3\mu^2(-1+\phi)+2(-1+\sigma)\sigma(-1+\phi)+\mu(-2-4\phi+\sigma(5+7\phi))\right)}{3H(-1+\sigma)\sigma^2(1+\phi)^2}<0.$$

The denominator is positive and so is $-2L\mu(-1+\phi)$. Hence, we end up with:

$$\beta=3\mu^2(-1+\phi)+2(-1+\sigma)\sigma(-1+\phi)+\mu(-2-4\phi+\sigma(5+7\phi))<0.$$

The second condition for stability of partial dispersions ($\xi<0$) implies:

$$\begin{aligned}
\frac{w_1}{w_2}\left(0,\tfrac{1}{2}\right) &< \frac{P_1^{\mu}}{P_2^{\mu}}\left(0,\tfrac{1}{2}\right) \Leftrightarrow \\
\frac{w_1}{w_2}\left(0,\tfrac{1}{2}\right) &< \left(\frac{\phi}{r}\right)^{\frac{\mu}{1-\sigma}}.
\end{aligned}$$

where:

$$\begin{aligned} \frac{w_1}{w_2} &= Dw_1 Dw_2^{-1} D D^{-1} \Leftrightarrow \\ \frac{w_1}{w_2} &= \frac{Dw_1}{Dw_2}. \end{aligned}$$

Beginning with region 1 we have:

$$Dw_1\left(0,\tfrac{1}{2}\right) = \frac{\mu}{\sigma}\frac{L}{3}\frac{1}{H}\left\{\frac{1}{\phi}+\frac{2\phi}{r}+\frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{1}{r^2}+\frac{\phi^2-1}{\phi r}\right]+\frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{1}{\phi}\frac{1}{4}\frac{1}{r^2}\right\}.$$

More tedious algebra renders:

$$Dw_1\left(0,\tfrac{1}{2}\right) = \frac{L\mu(-\mu+\sigma+(\mu+\sigma)\phi)(\sigma+\mu(-1+\phi)(1+2\phi)+\sigma\phi(1+4\phi))}{3H\sigma^3\phi(1+\phi)^2}.$$

We should rewrite Dw_2:

$$\begin{aligned} Dw_2\left(0,\tfrac{1}{2}\right) &= 1+\frac{\mu}{\sigma}\frac{\phi-1}{\phi+1} \Leftrightarrow \\ Dw_2\left(0,\tfrac{1}{2}\right) &= \frac{L\mu(-\mu+\sigma+(\mu+\sigma)\phi)}{H\sigma^2(1+\phi)}. \end{aligned}$$

It is now easier to obtain the quotient $\frac{Dw_1}{Dw_2}$:

$$\begin{aligned} \frac{Dw_1}{Dw_2}\left(0,\tfrac{1}{2}\right) &= \frac{L\mu(-\mu+\sigma+(\mu+\sigma)\phi)(\sigma+\mu(-1+\phi)(1+2\phi)+\sigma\phi(1+4\phi))H\sigma^2(1+\phi)}{3H\sigma^3\phi(1+\phi)^2L\mu(-\mu+\sigma+(\mu+\sigma)\phi)} \Leftrightarrow \\ \frac{Dw_1}{Dw_2}\left(0,\tfrac{1}{2}\right) &= \frac{\sigma+\mu(-1+\phi)(1+2\phi)+\sigma\phi(1+4\phi)}{3\sigma\phi(1+\phi)}. \end{aligned}$$

Finally, the condition $\xi < 0$ becomes:

$$\begin{aligned} \frac{\sigma+\mu(-1+\phi)(1+2\phi)+\sigma\phi(1+4\phi)}{3\sigma\phi(1+\phi)} &< \left(\frac{2\phi}{1+\phi}\right)^{\frac{\mu}{1-\sigma}} \Leftrightarrow \\ \sigma\phi(1+\phi)\left\{\sigma+\mu(-1+\phi)(1+2\phi)+\sigma\phi\left[1+4\phi-3\left(\frac{2\phi}{1+\phi}\right)(1+\phi)\right]\right\} &< 0 \quad \Leftrightarrow \\ \sigma+\mu(-1+\phi)(1+2\phi)+\sigma\phi\left[1+4\phi-3\left(\frac{2\phi}{1+\phi}\right)^{\frac{\mu}{1-\sigma}}(1+\phi)\right] &< 0. \end{aligned}$$

Hence, the partial dispersion equilibrium is stable if and only if:

$$\begin{cases} \xi < 0 \equiv \sigma + \mu(-1+\phi)(1+2\phi) + \sigma\phi \left[1 + 4\phi - 3\left(2\frac{\phi}{1+\phi}\right)^{\frac{\mu}{1-\sigma}}(1+\phi)\right] & < 0. \\ \beta < 0 \equiv 3\mu^2(-1+\phi) + 2(-1+\sigma)\sigma(-1+\phi) + \mu(-2-4\phi+\sigma(5+7\phi)) & < 0. \end{cases}$$

□

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